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GEOMETRICAL OPTICS

BY H. T. FLINT

WAVE MECHANICS

BY B. L. WORSNOP AND H. T. FLINT

ADVANCED PRACTICAL PHYSICS FOR STUDENTS

TRANSLATED BY H. T. FLINT

AN INTRODUCTION TO THE STUDY OF WAVE MECHANICS

GEOMETRICAL OPTICS

by

H. T. FLINT, PH.D., D.Sc.
Reader in Physics in the University of London, King's College

WITH 128 DIAGRAMS

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PREFACE

This book is written as a contribution to the teaching of Geometrical Optics, the method employed in the presentation of the subject having been tested during some years of experience of teaching University students both in the elementary and more advanced parts of this branch of their studies.

It must be admitted that this subject has suffered from a lack of agreement amongst teachers and other workers as to the best method of presenting it and from a discontinuity in the notation employed between the elementary and more advanced stages. The result has been neglect and distaste on the part of students who do not intend later on to specialize in this branch of their work.

This is very much to be regretted since the subject illustrates the principles of the description of natural phenomena with an elegance and simplicity which are not surpassed in any part of Physics. Moreover, the student uses optical apparatus from his earliest days in the laboratory and he should have a knowledge of the principles underlying its construction.

The purpose in writing this book has been to introduce the subject to the student interested in Physics in all its branches, but, at the same time, it is hoped that it will provide a good starting-point for one who may later on become a specialist in Optics.

The early chapters are not written in sufficient detail to make them a complete account for the elementary student. They are an introduction to the study of complex optical systems and are intended to bring to the

notice of teachers a method of presenting the subject which is simple and at the same time consistent and continuous with the requirements of its more advanced parts.

Shortly after the book was begun a report on the teaching of Geometrical Optics was published by the Physical Society. It appeared that some of the recommendations had already been followed, but some very helpful suggestions with regard to notation and to the use of the term " power " have been incorporated. It has not seemed necessary to lay stress on the convention with regard to the measurement of angles since that is already contained in our co-ordinate systems.

With regard to the presentation of the theory of collinear systems, we find ourselves in agreement with the very small minority on the committee which has found the presentation described in Drude's *Theory of Optics* satisfactory as a basis of teaching. We hope that the present work shows that the convention with regard to distances is suitable for elementary as well as for degree students. It would seem a very natural procedure to measure distances along the axis with opposite signs in the object and image systems. Students at the beginning of their course appreciate this if reference is made to the plane mirror. It is clear that after Alice had arrived at the other side of the looking-glass and found herself in the realm of the images, she was as inclined to measure *from* the mirror as she had been before.

Everyone will agree that, if the terms positive and negative are used, the former more naturally applies to converging systems. We believe that this should apply to the focal length in both object and image systems. For these reasons we have measured our positive distances along the axis in the two systems in opposite directions.

It is somewhat difficult to acknowledge adequately all the sources used in collecting the notes which have formed the basis of lectures and of this volume. But great indebtedness is due to Drude's *Theory of Optics*, to Lummer's *Photographic Optics*, to Martin's *Introduc-*

tion to Applied Optics and to the *Report of the Physical Society*.

It is also a pleasure to take this opportunity of expressing appreciation of the help given in conversations with Prof. Allan Ferguson, Mr. D. Orson Wood and with Dr. W. E. Williams with whom the author has been associated in teaching the subject at King's College.

H. T. F.

WHEATSTONE LABORATORY
KING'S COLLEGE
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CHAPTER I

INTRODUCTORY AND REFLECTION AT A PLANE MIRROR

Rays and Co-ordinates

Although light is regarded as consisting of waves, in the treatment of the branch of the subject known as geometrical optics we have no need to refer directly to waves. In our study of the subject in this book we shall make use of the idea of rays of light. In doing so we shall use a straight line to represent a ray, but it must be understood that this is a matter of convenience. It is impossible to isolate a single ray in this way by any practical means, and the nearest practical approach to this abstraction is a very narrow cone with its apex at the source of light. The straight line, which we draw to represent the ray, is the axis of this cone and may be called the principal ray. The use of rays in geometrical optics is bound up with the assumption of the rectilinear propagation of light, which is illustrated by the formation of shadows. The shadow cast by any plane object placed in the path of a cone of rays is similar to the object (fig. 1.1).

Light can only be considered to travel in straight lines if the length of the waves is small enough to be neglected in comparison with the lengths of the objects to be considered in the various problems.

In this limited region of the study of light we may describe the occurrences by means of rays starting out from the source of light. The rays are straight lines so long as the medium, in which the light travels, is homo-

geneous and they undergo changes of direction at reflecting surfaces and at surfaces separating different media.

This part of the study of light is described as geometrical optics, the term being used to describe the mathematical method adopted in describing the phenomena appropriate to this region. The part of the study of light to which the limitations here imposed do not apply is called physical optics.

The geometrical method which we shall use is analytical, and we shall employ the Cartesian system of co-ordinates.

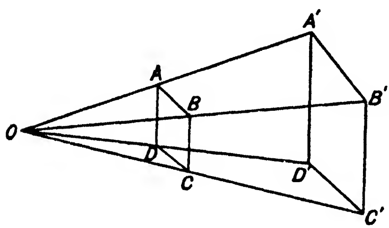

Fig. 1.1

It is well known that it is usual to choose a point as origin, and two straight lines at right angles to one another passing through this point as axes of co-ordinates, the particular origin and lines chosen being a matter of convenience.

Let us consider the example of the plane mirror as an illustration. Our ordinary experience of such mirrors teaches us that an object in front of the reflecting surface gives rise to an image behind it.

If this particular case is examined carefully, we discover that the image lies as far behind the surface as the object is in front, and that the image is of the same size as the object.

In this case it is clear that the simplest set of axes is obtained by drawing a line through the foot O of the object OP (fig. 1.2), perpendicularly to the reflecting surface, AB, cutting it in C and choosing OC and AC as axes.

We may then locate such points as O and O′ by means of abscissæ x and x', and P and P′ by means of co-

ordinates (x, y), (x', y'), measurements to the right and upward being taken as positive.

We could take different axes for the location of the image from those chosen for the location of the object. In the example just taken this would be an unnecessary complication. It would, however, be quite reasonable to choose measurements to the right as positive for the image, while giving the positive sign to measurements to the left in the case of the object.

If we imagine the image to be capable of making its own measurements, it would naturally make measurements from the reflecting surface positively towards the right, while the object would measure its distances from the mirror positively towards the left.

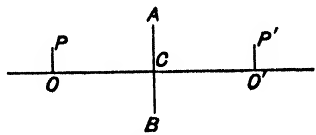

Fig. 1.2

We obtain in this way two co-ordinate systems differing in the convention with regard to sign along the x-axis. We can describe the two systems as the object and image system respectively, or as the object and image spaces. The use of the term " space " must not be allowed to lead to the confusion of thinking that the two spaces are distinct from one another. We use the term to indicate the system of co-ordinates.

We shall always use the convention that distances measured upward are positive and downward negative.

Also, we shall have no need to introduce a third co-ordinate, z, since all the cases we shall consider will be symmetrical about the x-axis.

We shall see later that it is not necessary and it is not always convenient to use the same origin for the object and image spaces. We shall always use the same line for the x-axis in the two cases, and this will be the line of symmetry. We shall take the y-axis as perpendicular to this axis.

Later in the book we shall see certain advantages in

adopting a particular set of axes and a particular convention to be applied to all the cases we consider, but the elegance of the subject will be missed if we confine ourselves to a special choice too soon.

Notation

If we refer to fig. 1.2, we see that the points O and P of the object space correspond to O′ and P′ of the image space. In the plane mirror the correspondence is very simple, but we shall find that a correspondence occurs in all cases of image formation. Such pairs of points as O, O′ and P, P′ are called conjugate points, and we shall denote this correspondence by using the same letter to denote the two points, but the image point will be distinguished by the dash.

We shall find that other quantities can be associated in pairs in this way, and we shall use the same device to denote this association.

The Laws of Reflection

When a ray of light falls on a reflecting surface PQ, whether plane or curved, its direction is changed in a definite way. To describe this change let the ray AM

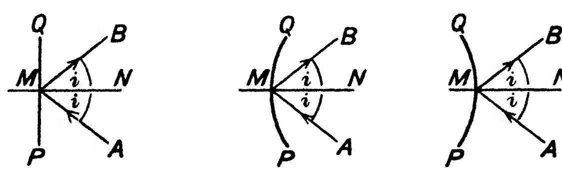

Fig. 1.3

strike the surface at M, and let MN be the normal to the surface at this point.

The ray is reflected along MB, and the laws of reflection are two in number.

(1) The incident ray (AM), the reflected ray (MB), and the normal (MN) to the surface lie in the same plane.

This is illustrated in the diagram which is drawn in the plane of the paper.

The plane determined by the incident ray and the normal is described as the plane of incidence, and the law states that the plane of incidence is the same as the plane of reflection which is determined by the normal and the reflected ray.

(2) The angle of incidence is equal to the angle of reflection; i.e. $\angle AMN = \angle BMN$.

These two angles are quantities which can be associated in the way described above, and will thus be denoted by i and i'. The second law gives the relation $i = i'$.

Reflection at a Plane Surface

We shall find that, as a result of reflection or refraction, rays originating from a point can be made either actually to pass through another point or to appear to proceed from another point. In such cases the first point is described as the object and the second point as the image. If the rays actually pass through the image, it is described as real. If the rays, when produced, pass through the second point, it is described as virtual.

An object may also be virtual; for rays directed to a point may fall on a reflecting surface before reaching it, and be reflected to pass through a second point. In this case the object is virtual and the image real. These points may be illustrated by considering reflection at a plane surface.

In fig. 1.4 the upper diagram shows a source of rays at O. It follows at once from the laws of reflection that after striking the plane mirror they will proceed as if they originated from O′. If they are received by an eye, it will appear that they come from O′, where OA = O′A. This is an example of a virtual image. The lower figure shows rays proceeding towards a point O, but falling on the mirror before reaching it, the laws of reflection show that after striking the mirror they will pass through O′. In this case we have a virtual object.

The Deviation of a Ray by a Plane Mirror

A ray OM (fig. 1.5) falling on a mirror at an angle of incidence i is deviated through the angle QMP to pass

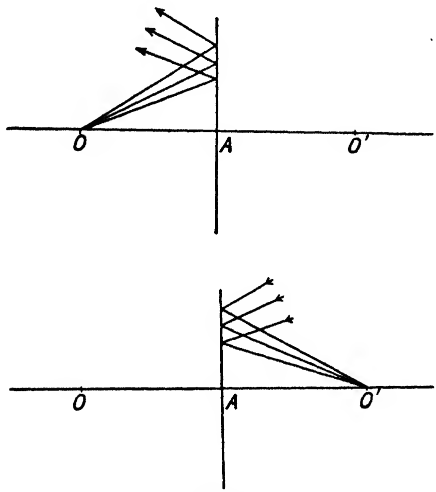

Fig. 1.4

along MP. The amount of the deviation is thus $(\pi - 2i)$.

A case of deviation by a plane mirror which is exemplified in many instruments is illustrated in fig. 1.6.

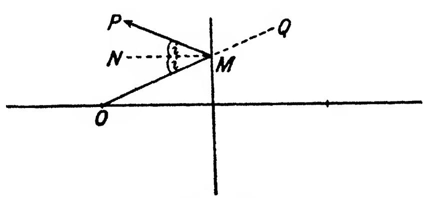

Fig. 1.5

A ray OM falls on a mirror and is deviated along MP. The mirror rotates through an angle A, and the ray now passes to M′ and is deviated along M′P′. It is required

to find the angle between MP and M′P′ in terms of the angle of rotation of the mirror.

Let the angle of incidence at M be i, then since the mirror and, consequently, its normal rotate through the angle A, the angle of incidence at M′ is $(i + A)$. The deviation of MP from the original direction of the ray is

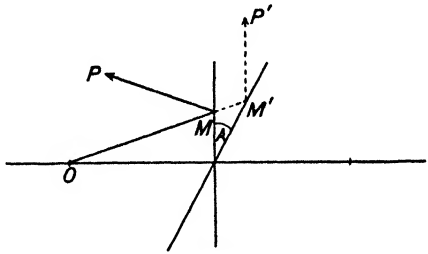

Fig. 1.6

$(\pi - 2i)$, and the deviation of M′P′ is $\{\pi - 2(i + A)\}$. Thus the angle between MP and M′P′ is 2A.

In many instruments a mirror and a narrow beam of light are used to record a deflection instead of a pointer. This system has the advantage that while the pointer, like the mirror, records the deflection A, the beam of light incident on the mirror will record a magnified deflection 2A. The principle is also used in the sextant.

Deviation of a Ray by Inclined Mirrors

In fig. 1.7 a ray OA is shown reflected at two mirrors AE and BE, inclined at an angle θ. The ray is deviated from an original direction OA to a final direction BD, i.e. through the angle ACD. Let the normals at A and B meet in F. It is clear that $\angle$AFB is equal to $(\pi - \theta)$, i.e. $(i + i') = \theta$, where i and i' are the angles of

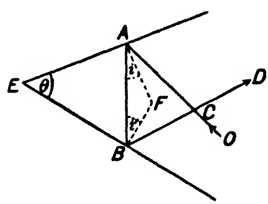

Fig. 1.7

incidence at the mirrors. But $\angle ACD = \angle CAB + \angle CBA = 2(i+i')$. Thus the deviation is 2θ. This is an interesting result because it depends only on the angle between the mirrors and not upon the angles of incidence, so that, if the mirrors rotate together through an angle, the amount of deviation is the same.

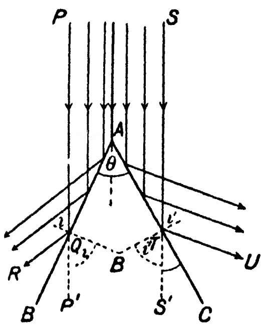

Fig. 1.8

A second case is that illustrated in fig. 1.8, where parallel rays fall on two reflecting surfaces AB and AC. It is required to find the angle between the rays reflected from the two mirrors, which, as shown, are inclined at an angle θ.

The angle is given by the sum of the deviation of the rays falling on AB and on AC. It is thus $\{2\pi - 2(i+i')\}$. It will be seen at once from the figure that this is equal to 2θ, for $\angle BQP' + \angle CTS' = \theta$ and $BQP' = \frac{\pi}{2} - i$, $CTS' = \frac{\pi}{2} - i'$.

This principle is often applied to measure the angle of a prism. Parallel rays from the collimator of a spectrometer are allowed to fall on two sides of a prism with its apex angle in the position of fig. 1.8. By means of the telescope of the instrument the rays are received after reflection at one side AB of the prism, and the telescope is then moved to receive the rays from AC. The instrument is provided with a scale for measuring the angle through which the telescope is turned, and one-half of this angle measures the angle of the prism.

Multiple Reflections in Inclined Mirrors

When a bright point is situated between two mirrors inclined to one another, a set of images is formed

in each mirror. In fig. 1.9 O denotes the point source of the rays and O′ is the image of O in AC. After reflection the rays OP and OS appear to come from O′. Thus, when the rays PQ and ST fall on BC, it is exactly as if they originated from an object situated at O′. In the figure the arc OA = AO′ and O′B = BO″. Thus the images in one mirror can be regarded as objects for the other mirror provided that they lie in front of the appropriate reflecting surface; but as soon as an image falls behind both mirrors, that is, within the arc A′B′, it can give rise to no further reflections.

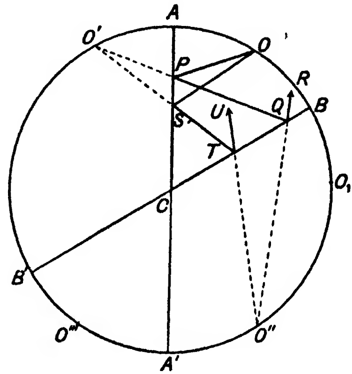

Fig. 1.9

We shall proceed to determine the number of images which can be formed in this way.

We have considered only one-half of the problem, because an image O_1 will be formed in BC, and this will give rise to further images.

Denote the angle between the mirrors by C and let the angle OCA be θ. It is clear that all the images lie on a circle with centre C and radius CO. Let the radius of the circle be a. Then

$$\text{arc OA} = \text{AO}^{\text{I}} = a\theta.$$
$$\text{arc O}'\text{B} = \text{BO}^{\text{II}} = a(\theta + \text{C}).$$

We have thus, measuring distances along the circumference from O,

$$\text{OO}^{\text{I}} = 2a\theta,\ \text{OO}^{\text{II}} = \text{OB} + \text{BO}^{\text{II}} = a(\text{C} - \theta) + a(\theta + \text{C}) = 2a\text{C},$$
$$\text{OO}^{\text{III}} = \text{OA} + \text{AO}^{\text{III}} = \text{OA} + \text{AO}^{\text{II}} = 2\text{OA} + \text{OO}^{\text{II}} = 2a\theta + 2a\text{C}.$$

If we proceed in this way step by step, we obtain the series

$$\begin{aligned} &OO^{\mathrm{I}} = 2a\theta, \quad OO^{\mathrm{III}} = 2a(\theta + C), \quad OO^{\mathrm{V}} = 2a(\theta + 2C), \text{ \&c.} \\ &OO^{\mathrm{II}} = 2aC, \quad OO^{\mathrm{IV}} = 4aC, \qquad\qquad OO^{\mathrm{VI}} = 6aC, \text{ \&c.} \end{aligned}$$

This applies to the first series of images arising from the first reflection in CA. For the second series we have similar results, θ being replaced by θ', where $\angle OCB = \theta'$. The series stops when an image falls on arc A′B′.

Suppose this is an image with an even number, $2n$, i.e. one of those which lie behind BC in the figure, then the series will stop if arc $OO^{2n} >$ arc OA′

$$\text{arc } OO^{2n} = 2naC \quad \text{and} \quad \text{arc } OA' = a(\pi - \theta).$$

Thus the series stops when:

$$2naC > a(\pi - \theta)$$

or

$$2n > \frac{\pi - \theta}{C}.$$

Suppose the number of the image is odd, thus lying behind AC in the figure. Then if it is denoted by O^{2n+1}, this point lies on A′B′ and arc $OO^{2n+1} >$ arc OAB′, or

$$2a(\theta + nC) > a(\pi - \theta').$$

But $\theta + \theta' = C$, thus

$$2a(\theta + nC) > a(\pi + \theta - C),$$

or

$$(2n + 1)C > \pi - \theta,$$

or

$$2n + 1 > \frac{\pi - \theta}{C}.$$

Thus whether the number is odd or even, as soon as it is greater than $\dfrac{\pi - \theta}{C}$, further formation of images ceases.

To illustrate this, consider two mirrors inclined at 45° with the angle θ of magnitude 20°. Then $\dfrac{\pi - \theta}{C} = 3\frac{5}{9}$.

Thus the fourth image will fall behind both mirrors and the number of images is four.

For the second series of images we must replace θ by θ', and the number of images in this series is given by the integer next above $\frac{\pi - \theta'}{C}$. In the example just quoted θ' is $25°$, and the value of $\frac{\pi - \theta'}{C}$ is $\frac{155}{45}$ or $3\frac{4}{9}$, the number being in this case also four.

The value of $\frac{\theta}{C}$ or of $\frac{\theta'}{C}$ is necessarily fractional, so that if $\frac{\pi}{C}$ is a whole number, in other words, if π is an integral multiple of the angle between the mirrors, the number of images formed will be $\frac{\pi}{C}$.

To illustrate this case, suppose the mirrors are inclined at $60°$, and let $\theta = 20°$, $\theta' = 40°$. The number of images in the first series is thus $\frac{180}{60}$ or 3, and similarly for the second series.

We have an odd number of images, the last of the series being O^{III} and O_{III} respectively.

According to our formulæ,

$$OO^{III} = 2a(\theta + C) = 2a\left(\frac{\pi}{9} + \frac{\pi}{3}\right) = 2a \cdot \frac{4\pi}{9},$$

$$OO_{III} = 2a(\theta' + C) = 2a\left(\frac{2\pi}{9} + \frac{\pi}{3}\right) = 2a \cdot \frac{5\pi}{9}.$$

These arcs are measured in opposite directions round the circle from O, and it will be noted that $OO^{III} + OO_{III} = 2\pi a$, or that O^{III} and O_{III} are the same points. Thus the last images of each series coincide and the total number is reduced by unity. The total number of images is thus $\left(\frac{2\pi}{C} - 1\right)$, or, including the radiating point itself, the number is $\frac{2\pi}{C}$.

This result, which we have illustrated in the special case of mirrors inclined at 60°, is true generally when C is a submultiple of π, for the last images of each series are always coincident in this case.

For suppose that $\frac{\pi}{C}$ is an even number, $2n$, then the last images are O^{2n} and O_{2n}, and

$$OO^{2n} + OO_{2n} = 2naC + 2naC = 2\pi a.$$

If $\frac{\pi}{C}$ is an odd number $(2n + 1)$, the last images are O^{2n+1} and O_{2n+1}, and

$$\begin{aligned} OO^{2n+1} + OO_{2n+1} &= 2a(\theta + nC) + 2a(\theta' + nC) \\ &= 2(2n + 1)aC = 2\pi a, \end{aligned}$$

since $\theta + \theta' = C$.

Thus the sum of the arcs in either case makes up the complete circumference or the two images are coincident.

CHAPTER II

REFLECTION OF RAYS FROM POINT SOURCES BY SPHERICAL MIRRORS

In this chapter we shall consider the reflection of rays, originating from point sources, by spherical surfaces. We shall limit ourselves to the particular case where the source lies on the axis of the surface. The reflecting surface forms part of a sphere and the axis is the line drawn through the centre of the sphere to the centre of the reflecting surface (CA in fig. 2.1). We can consider the reflecting surface as generated by the circular arc AP as it rotates about the axis CA.

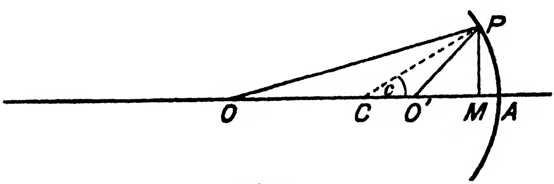

Fig. 2.1

The surface may be polished on either side, i.e. the mirror may be concave or convex. If polished on the side of the centre of curvature the mirror is concave, and if polished on the other side it is convex.

In the figure C denotes the centre of curvature, and the point A is described as the pole of the mirror. The length CA is the radius of the sphere of which the mirror is a part, and it will be described as the radius of curvature of the surface. We shall distinguish concave and convex surfaces by giving the negative sign to this radius

for the former and the positive sign for the latter. Thus if we describe a mirror as having a radius of -10 cm., we mean that it is concave and the radius of the sphere is 10 cm. The formulæ we shall develop will take account of this convention, and in using them we must use the appropriate sign for the radius of curvature; on the other hand, if a calculation by means of the formula gives a negative value for the radius of curvature, we know that the mirror is a concave one. If the result of the calculation is a positive radius, the mirror is convex. We must regard this merely as a convention; it is one of the conventions we shall finally introduce for convenience in making calculations. This particular convention with regard to the sign of the radius of curvature must not be confused with others to be introduced later; it is quite independent. There is no reason, except that of convenience, for applying the negative sign to the concave mirror; we might have adopted the opposite system of signs.

We have still to make a choice with regard to the origin and the system of co-ordinates, but we shall leave this question open until later.

The Concave Spherical Mirror

We shall begin with a consideration of the concave spherical mirror, but some of the remarks will be of a general character.

We know from experience some of the properties of this reflecting surface. If a large object is placed in front of it, we see, on looking into the mirror, a distorted picture of the object. But if the object is small and lies on or close to the axis, we obtain a picture with none or very little distortion. The picture is a true one, though it may differ in size from the original.

We shall not now be concerned with cases where distortion occurs, and, for the present, we shall consider reflection of rays from points on the axis, e.g. O in fig. 2.1.

Let a ray OP strike the surface at P.

The first law tells us that the reflected ray lies in the plane of OP and the normal, CP, i.e. in the plane of the paper, and the second law that the angles of incidence and reflection are equal. Let the reflected ray cut the axis in O′, then the angles OPC and O′PC are equal.

We shall suppose that the ray OP makes a small angle with the axis. In saying this we mean that the angle, say θ, is small enough to make it possible to write

$$\sin\theta = \theta, \quad \cos\theta = 1, \quad \tan\theta = \theta \quad . \quad . \quad 2.1$$

without appreciable error, the angle being, of course, measured in radians. The reader may examine to what extent this is true by reference to tables; the following are the values for 1° and 5°

$$1^\circ = \cdot 0175 \text{ radians}, \quad \sin 1^\circ = \cdot 0175, \quad \tan 1^\circ = \cdot 0175$$
$$5^\circ = \cdot 0873 \text{ radians}, \quad \sin 5^\circ = \cdot 0872, \quad \tan 5^\circ = \cdot 0875.$$

The validity of the formulæ to be developed will depend upon the accuracy with which the relations (2.1) apply, and rays to which this approximation may be applied are described as paraxial. We are thus limited to rays falling close to A.

From the figure

$$MC = PC \ \cos C = PC,$$

since C is a small angle.

We can thus write MC = PC = AC, or regard A and M as coincident. These equalities mean that the lengths concerned do not differ from one another by amounts greater than those of the order C^2.

Let us choose A as origin, and measure distances along the axes positively towards the left, placing the object on the left of A. The light thus travels towards the negative direction. Let AO be denoted by l, AO′ by l', and let us proceed to find a relation between l and l'.

The Formula for the Concave Mirror

In the triangle OPO′, CP is the internal bisector of the angle OPO′. Hence

$$\frac{OP}{O'P} = \frac{OC}{CO'}. \qquad \ldots \qquad 2.2$$

On account of the smallness of the inclinations of the lines OP, CP, and O′P to the axis, we can write 2.2 in the form

$$\frac{OA}{O'A} = \frac{OC}{CO'}$$

or

$$\frac{OA}{O'A} = \frac{OA - AC}{AC - O'A}.$$

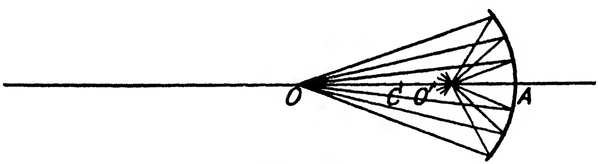

Fig. 2.2

Let r denote the radius of curvature of the surface, i.e. in accordance with the convention, $r = -AC$. Then

$$\frac{l}{l'} = \frac{l + r}{-r - l'}$$

or

$$\frac{1}{l'} + \frac{1}{l} = -\frac{2}{r}. \qquad \ldots \qquad 2.3$$

We have obtained a formula which holds for all rays such as OP, provided that the angle AOP is small.

Thus all rays from O satisfying this condition will pass through O′ (fig. 2.2) after reflection. The mirror

is said to focus the rays at O′, and O and O′ are conjugate points.

The direction of the rays may be reversed or, if O′ is the source of radiation, O will become the image. This is also seen from the equation (2.3), for it is still valid if l and l' be interchanged.

We must not expect the rays from O to pass through O′ if they are inclined to the axis at large angles. We shall see that the mirror will not in general focus the rays in such cases.

Self-conjugate Points

The points A and C have the property that if an object is placed there, the image is formed in the same place.

In the case of A this is a triviality, for when an object is placed on a mirror the image is seen also at the mirror, or on bringing up an object close to the mirror the image can be seen approaching from behind until they come together at the surface.

In the case of C it is clear that rays from this point strike the reflecting surface normally and are directed back to pass through C again. This fact is made use of in finding the radius of curvature of a concave mirror. A pin point is moved along the axis until the image is seen coinciding with the object. The pin then lies at the centre of curvature.

In the cases we shall consider first, there is a point such as A which lies on the reflecting or refracting surface and possesses this self-conjugate property. It is for this reason that A is chosen as the origin of the co-ordinate system. The self-conjugate property of the origin gives to the formulæ a simple form. This point will be more fully explained as we proceed.

Focal Points

If the object O is very distant, i.e. if l is large,

$\frac{1}{l}$ becomes very small, and in the limit vanishes. To find the value of l' in this case, we have (2.3)

$$\frac{1}{l'} = -\frac{2}{r}$$

$$l' = -\frac{1}{2}r.$$

The value of l' is positive, since in the case of a concave mirror r is negative. If the mirror has a radius 10 cm., we have to write in the formula $r = -10$. Hence in this case $l' = 5$ cm.

We see that the image of a distant object lies midway between the pole and centre of curvature of the mirror,

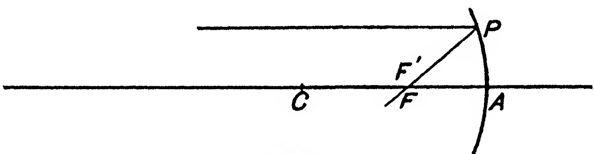

Fig. 2.3

or in other words, parallel rays come to a focus at this point, for the rays from the distant object may be regarded as parallel amongst themselves and to the axis. This point is denoted by F′ in fig. 2.3, and is often called the principal focus of the mirror. We shall describe it as the image focal point, to denote the fact that it is the image corresponding to parallel rays.

It is also interesting to consider the case when the image lies at a great distance, or when the reflected rays are parallel. It is clear from the fact that the direction of the light along the rays can be reversed that when an object is placed at F′ the reflected rays will be parallel. The point is thus also an object focal point, and in keeping with our notation it should be denoted by F. We shall see later that object and image focal points do not always coincide. It happens in this case that the two

points are coincident because of the simple geometry of the surface.

We have laid stress on this point because of its importance later on in our work, and have attached the letters F and F′ to the point in the diagram (2.3).

The distance from A to the focal point is called the focal length. AF is the object focal length, denoted by f, and AF′ is the image focal length, denoted by f'. In the present example

$$f = f' = -\tfrac{1}{2}r. \qquad . \quad . \quad . \quad . \quad 2.4$$

We note that the focal length has been measured from A, and this point is chosen principally because of its property of being self-conjugate. We could call it the principal point in anticipation of a term we shall introduce later on.

We notice that (2.3) can be written in the form

$$\frac{f'}{l'} + \frac{f}{l} = 1, \qquad . \quad . \quad . \quad . \quad 2.5$$

where f and f' have the value $-\tfrac{1}{2}r$ (2.4).

This form of the equation is introduced because, as we shall see later, it applies to all optical systems, mirrors, lenses, and combinations of lenses such as occur in eyepieces, each particular reflecting or refracting instrument being characterised by its special values of f and f'.

In the case now considered these values are given by (2.4).

An Alternative System of Co-ordinates

We shall now examine the form taken by (2.3) if we alter the system of co-ordinates. As in the last case, we shall measure object distances from A positively to the left. We shall, however, measure image distances from A positively towards the right. In our diagrams this

means that image distances are measured positively in the direction of the rays and object distances negatively in this direction.

For a concave mirror this system may seem impractical, but we have seen in the case of the plane mirror that such a system seems, from one point of view, a natural one to choose.

In other cases we shall see that this scheme has much to recommend it.

To distinguish this system of measurement from the last, let the object distance be denoted by L and the image distance by L′.

In fig. 2.1 we have

$$L = AO, \ L' = -AO'.$$

From (2.2) we deduce as before:

$$\frac{OA}{O'A} = \frac{OA - AC}{AC - O'A},$$

or

$$\frac{L}{-L'} = \frac{L + r}{-r + L'},$$

or

$$\frac{1}{L'} - \frac{1}{L} = \frac{2}{r}. \qquad \ldots \quad 2.6$$

When L is very great, $L' = \frac{1}{2}r$, and when L′ is very great, $L = -\frac{1}{2}r$.

Thus the focal lengths in this system are $f' = \frac{1}{2}r$, $f = -\frac{1}{2}r$.

We see that since r is negative, the value of L′ is negative, which means in our present system that the image focal point lies to left of A, viz. at F′. The value of L is in the same way positive and lies to the left of A, viz. at F. This must be so, for no change in the system of co-ordinates can alter the focal points, only the method

of describing them can be changed. We described the focal lengths by

$$f = f' = -\tfrac{1}{2}r$$

in the last case.

We now describe the focal lengths by

$$f = -\tfrac{1}{2}r, \quad f' = \tfrac{1}{2}r. \qquad . \quad . \quad . \quad 2.7$$

Both must mean the same thing, although described in a different language. Both methods of description mean that the focal point lies at F (or F′) midway between A and C. But the remarkable thing to note is that (2.6) can be written in the form

$$\frac{\frac{1}{2}r}{L'} - \frac{\frac{1}{2}r}{L} = 1,$$

or making use of the values given in (2.7), which are appropriate to this case

$$\frac{f'}{L'} + \frac{f}{L} = 1.$$

This is identical in form with (2.5). The system of co-ordinates has been changed, and with it the values of f and f' undergo a change, viz. (2.7) instead of (2.4), but the form of the relation between the conjugate distances remains the same, viz.

$$\frac{f'}{l'} + \frac{f}{l} = 1. \qquad . \quad . \quad . \quad . \quad . \quad . \quad 2.5$$

We see, however, that the value of f and f' change with the change of convention, and it will therefore be an advantage ultimately to decide on the most convenient system and to employ it in all cases.

Change of Origin

We shall now examine the form of the equation for l and l' when the measurements are made from C, that is to say, we shall change the origin from A to C.

Let us make our measurements positively to the left for the object and positively to the right for the image.

Thus in fig. 2.1, $l = \text{CO}$, $l' = \text{CO}'$. Note that we have altered the meanings of l and l' from that used previously.

We have (2.2)

$$\frac{\text{OP}}{\text{O}'\text{P}} = \frac{\text{OC}}{\text{CO}'}.$$

Hence

$$\frac{\text{OC} + \text{CA}}{\text{CA} - \text{CO}'} = \frac{\text{OC}}{\text{CO}'},$$

$$\frac{l - r}{-r - l'} = \frac{l}{l'},$$

whence

$$\frac{1}{l'} - \frac{1}{l} = -\frac{2}{r}.$$

The distances of the foci are given by

$$\frac{1}{l'} = -\frac{2}{r} \quad \text{and} \quad \frac{1}{l} = \frac{2}{r}.$$

Let these values of l and l' be denoted by f and f'. Then

$$f = \tfrac{1}{2}r, \; f' = -\tfrac{1}{2}r.$$

These values show again that the foci are coincident and situated midway between A and C, and (2.6) once more takes the form

$$\frac{f'}{l'} + \frac{f}{l} = 1,$$

where f and f' are lengths measured from C to the principal focus.

The preservation of the form of the equation in this case is due to the fact that C is a self-conjugate point. These illustrations serve to show the importance of this form of the equation, its independence of the sign convention, and also to some extent its independence of

choice of origin. We have seen that when the origin is self-conjugate the same form is preserved; later on we shall be able to extend this property still further.

Origin at the Focal Point

It is important to examine the equation obtained when the focal point F(F′) is taken as origin. It is usual in this case to denote the distances by x and x'. Referring to fig. 2.4, we have

$$x = \mathrm{FO}, \quad x' = \mathrm{F'O'},$$

where we are now measuring positively to the left in both cases.

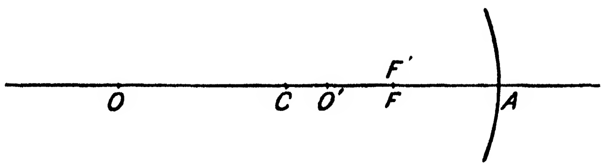

Fig. 2.4

From (2.2) we can deduce

$$\frac{\mathrm{OA}}{\mathrm{O'A}} = \frac{\mathrm{OC}}{\mathrm{CO'}}.$$

Hence

$$\frac{\mathrm{OF} + \mathrm{FA}}{\mathrm{O'F} + \mathrm{FA}} = \frac{\mathrm{OF} - \mathrm{CF}}{\mathrm{CF} - \mathrm{O'F}}.$$

Remembering that $\mathrm{FA} = \mathrm{FC} = -\frac{1}{2}r$, with the sign convention for r, we have

$$\frac{x - \frac{1}{2}r}{x' - \frac{1}{2}r} = \frac{x + \frac{1}{2}r}{-\frac{1}{2}r - x'}.$$

It follows that

$$xx' = (\tfrac{1}{2}r)^2.$$

On introducing the values of f and f' appropriate to this co-ordinate system (2.4), we find

$$xx' = ff'. \qquad \ldots \ldots \qquad 2.8$$

We have introduced this form because this is the general equation which, as we shall see later, is applicable to all optical systems.

We can satisfy ourselves of its independence of the system of co-ordinates by proceeding as before, taking measurements to the right as positive for x'. It will be found that the form (2.8) is still preserved, f and f' now being given by (2.7).

The Convex Spherical Mirror

The treatment in the case of the convex mirror is similar to that in the former case. It will be noticed that for any object O (fig. 2.5), lying to the left of A, the image lies to the right of A, i.e. behind the mirror, and is virtual. The image will be real if the rays are directed to a point behind the mirror, i.e. the object is virtual. Fig. 2.5 will illustrate this if the ray OPQ be reversed in direction and O and O′ interchanged.

We shall use in this case the opposite directions for object and image measurements

$$l = \mathrm{AO}, \quad l' = \mathrm{AO}',$$

and in this case r is positive and equal to AC. CP bisects the vertical angle of the triangle externally, and we have

$$\frac{\mathrm{OP}}{\mathrm{O'P}} = \frac{\mathrm{OC}}{\mathrm{O'C}}. \qquad \ldots \ldots \qquad 2.9$$

Proceeding exactly as before, we obtain

$$\frac{\mathrm{OA}}{\mathrm{O'A}} = \frac{\mathrm{OA} + \mathrm{AC}}{\mathrm{AC} - \mathrm{O'A}}$$

or

$$\frac{l}{l'} = \frac{l + r}{r - l'},$$

whence

$$\frac{1}{l'} - \frac{1}{l} = \frac{2}{r}. \quad . \quad . \quad . \quad . \quad . \qquad 2.10$$

The focal lengths are given by $\frac{1}{l'} = \frac{2}{r}$, $\frac{1}{l} = -\frac{2}{r}$, and denoting these lengths by f' and f, we have

$$f' = \tfrac{1}{2}r,\ f = -\tfrac{1}{2}r.$$

Thus f' is positive and f negative, i.e. F′ lies to the right and F to the right of A, both being coincident at the middle point of CA.

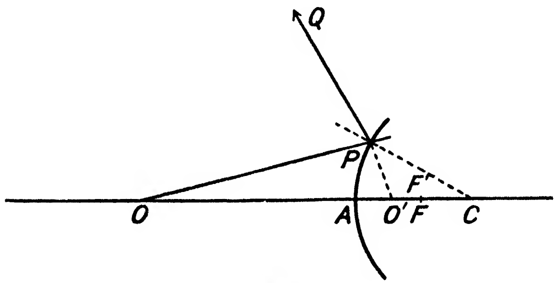

Fig. 2.5

The equation can once more be put in the form

$$\frac{f'}{l'} + \frac{f}{l} = 1,$$

with

$$f = -\tfrac{1}{2}r,\ f' = \tfrac{1}{2}r. \quad . \quad . \quad . \qquad 2.11$$

If this be compared with (2.7), which gives the values for the concave mirror, with the same sign convention, we see that the results are identical.

Thus (2.7) or (2.11) is a general equation applicable to both kinds of mirrors. In applying the values we have merely to take account of the difference in the sign of r.

In the same way as before, we can obtain the formula $xx' = ff'$ for the case of the convex mirror.

We can thus sum up the formulæ applicable to mirrors, as follows

$$\frac{f'}{l'} + \frac{f}{l} = 1, \quad \dots \dots \quad 2.5$$

$$xx' = ff', \quad \dots \dots \quad 2.8$$

and if the sign convention is positive to the left for the object, and positive to the right for the image, the pole of the mirror being the origin, and the direction of the ray being along the positive direction of the image space,

$$f = -\tfrac{1}{2}r, \; f' = \tfrac{1}{2}r. \quad \dots \quad 2.11$$

The Relative Positions of Object and Image

(*a*) *The Concave Mirror*

The simplest treatment of this question is obtained by using the formula

$$xx' = ff'.$$

On inserting the values (2.11) for f and f',

$$xx' = -\tfrac{1}{4}r^2.$$

The right-hand side is thus negative.

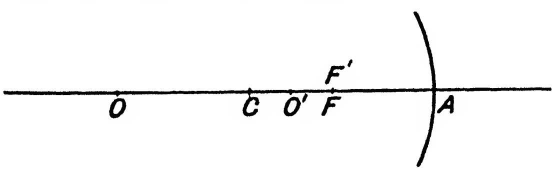

Fig. 2.6

When the object is to the left of F, x is positive and therefore x' is negative, i.e. the image O′ is also to the left of F (fig. 2.6).

When $x > \frac{1}{2}r$, i.e. the object lies beyond the centre of curvature, $x' < \frac{1}{2}r$, i.e. the image lies between F and C.

The converse, that if the object lies between F and C, the image lies beyond C, is similarly true.

If the object lies between A and F, x is negative and numerically less than $\frac{1}{2}r$. Thus x' is positive and numerically greater than $\frac{1}{2}r$. In other words, the image lies to the right of A and is thus virtual.

These results are of importance in practical work, where it is necessary to determine pairs of conjugate points. The position of the centre of curvature and the focal point should first be found approximately. Then

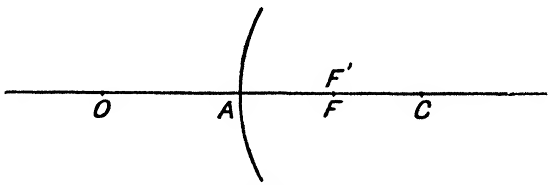

Fig. 2.7

if the object is set up beyond C, it is only necessary to explore between F and C to find the position of the image. If the object is placed between A and F, much time will be lost in the vain search for a real image.

(*b*) *The Convex Mirror*

The same formula is applicable here, viz.

$$xx' = -\tfrac{1}{4}r^2.$$

If x is positive and greater than $\frac{1}{2}r$, as is the case for all real objects (fig. 2.7), x' will be negative and less than $\frac{1}{2}r$, i.e. the image O′ will lie to the left of F, within FA. The image is thus virtual.

CHAPTER III

REFRACTION OF RAYS FROM POINT SOURCES BY PLANE AND SPHERICAL SURFACES

Refraction at the Boundary of two Media

When a ray of light passes from one homogeneous medium to another, the ray is a straight line in both media, but there is a change of direction at the boundary. This is illustrated in fig. 3.1, the boundary being represented as plane in one case and curved in the other.

There are two laws which describe the passage from a medium A to another medium A′.

(1) The incident ray PQ, the normal to the bounding surface NN′, and the refracted ray QP′ lie in one plane.

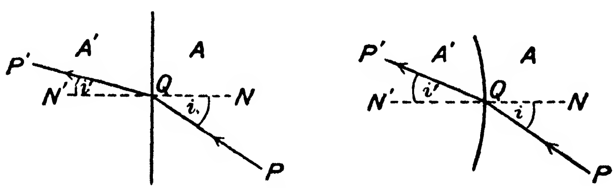

Fig. 3.1

The plane defined by the incident ray and the normal is described as the plane of incidence, and that defined by the refracted ray and the normal as the plane of refraction. The first law states that these planes are identical.

(2) The ratio of the sine of the angle of incidence i to the sine of the angle of refraction i' is constant for a particular pair of media. Reference may be made to

fig. 3.1. The meaning of this is that the ratio does not depend upon the magnitude of the angles i and i'.

We can write the second law in the form

$$\frac{\sin i}{\sin i'} = n_{AA'}. \qquad \qquad 3.1$$

$n_{AA'}$ is described as the refractive index for the two media, and the order of the suffixes AA′ denotes that the ray is travelling from A to A′. If the ray is travelling in the reverse direction, the index is denoted by $n_{A'A}$. The path of light is reversible, that is to say, if a ray passes along PQ and takes the direction QP′ in the second medium, then a ray along P′Q will take the direction QP on emerging into the medium A. Thus

$\dfrac{\sin i'}{\sin i} = n_{A'A}$ in accordance with the definition of $n_{A'A}$,

or

$$n_{AA'} = \frac{1}{n_{A'A}}. \qquad \qquad 3.2$$

To take an example in the passage of a ray from air to water where the refractive index is $\frac{4}{3}$, we see from (3.2) that the refractive index from water to air is $\frac{3}{4}$.

A ray of light may be bent nearer to the normal on passing into the second medium, in which case the second medium is said to be optically denser than the first. In such cases the refractive index is greater than unity. When the converse is the case, the second medium is optically less dense. Suppose a ray

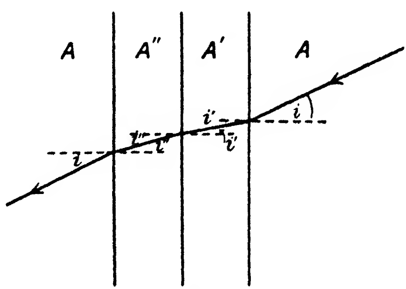

Fig. 3.2

passes from air A (fig. 3.2) through two media A′ and A″, bounded by parallel planes, and out into air again. It is found by experiment that the ray emerging into the air is parallel to the original ray. By referring to the figure, it may be seen that

$$\frac{\sin i}{\sin i'} = n_{AA'}, \quad \frac{\sin i'}{\sin i''} = n_{A'A''}, \quad \frac{\sin i''}{\sin i} = n_{A''A}.$$

Thus

$$n_{AA'} n_{A'A''} n_{A''A} = 1, \qquad . \quad . \quad . \quad . \quad . \quad 3.3$$

or

$$n_{A'A''} = \frac{1}{n_{AA'} n_{A''A}} = \frac{n_{AA''}}{n_{AA'}}. \qquad . \quad . \quad . \quad . \quad 3.4$$

To take a definite example, let the medium A′ be glass and A″ water. The equation (3.4) shows that if the refractive indices with respect to air are known for the two media, we can determine the index from one to the other.

Thus since the refractive index from air to water is $\frac{4}{3}$, and from air to glass $\frac{3}{2}$, we can deduce that the refractive index from glass to water is $\frac{\frac{4}{3}}{\frac{3}{2}}$ or $\frac{8}{9}$.

It is thus convenient to make use of air as a standard, and to measure refractive indices of other substances with respect to it. In accurate work the standard of reference is a vacuum, but the refractive index of air differs only slightly from unity.

Thus when we speak of the refractive index of a medium, we assume that it is referred to air or a vacuum. Refractive indices with respect to other media can be obtained by means of (3.4). We shall use the letter n to denote refractive indices, with a suffix if necessary.

Returning to (3.1), we can write $n_{AA'} = \frac{n_{A'}}{n_A}$ or simply $\frac{n'}{n}$, where $n_{A'}$ or n' means the refractive index for the medium

A′ with respect to a vacuum, and n that for A. Thus

$$\frac{\sin i}{\sin i'} = \frac{n'}{n},$$

or

$$n \sin i = n' \sin i'. \quad . \quad . \quad . \quad . \quad 3{\cdot}5$$

Thus the product $n \sin i$ does not change as we pass from one medium to another. This is another way of stating the second law of refraction.

Critical Reflection

When a ray of light passes from a denser to a lighter medium, the ratio $\frac{\sin i}{\sin i'}$ is less than unity, e.g. in the case of a ray passing from glass to air this ratio is $\frac{2}{3}$. The angle i' is always greater than i, and the interesting question is: What happens when i' has increased to a right angle? (Fig. 3.3.) When i' is a right angle, we have

$$\sin i = n_{ga} = \frac{1}{n}. \quad 3.6$$

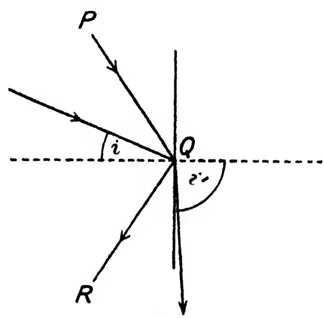

Fig. 3.3

n_{ga} denotes the refractive index from the medium (say glass) to air, and n is the refractive index of glass.

The particular value of i given by (3.6) is called the critical angle for glass.

When the angle of incidence is greater than i, the ray is totally reflected (PQR), and no light from this direction gets into the second medium. For angles less than i there is partial reflection as well as refraction, but beyond this critical angle there is no refraction at all.

Rays at Small Inclination to the Normal

Let PN (fig. 3.4) denote the plane separating a denser medium on the left from air.

Let O denote a point source of rays in the denser medium on the normal ON. Let a ray from O strike the plane at P and be refracted out into the air, the angle of incidence i being small. The emergent ray will be bent upward in the figure and will appear to proceed from O′.

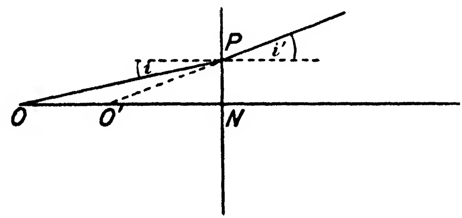

Fig. 3.4

We have $\dfrac{\sin i}{\sin i'} = \dfrac{1}{n}$, or, since the angles are small,

$$ni = i'.$$

Again, since the angles are small

$$i = \tan i = \frac{\text{PN}}{\text{ON}} \quad \text{and} \quad i' = \frac{\text{PN}}{\text{O'N}}.$$

Thus

$$\frac{\text{ON}}{\text{O'N}} = n. \qquad \ldots \ldots \quad 3.7$$

This is true for all rays proceeding from O at small angles. Thus O′ is the image of O, and an eye looking from the air on the right of the boundary will see the object O apparently at O′, nearer to the surface.

This is exemplified by pools of water, which always appear shallower than is actually the case.

Refraction by Prisms

A very important case of refraction, on account of its application in experimental work, is that through a prism. A prism may be described as a portion of a medium bounded by two plane faces meeting at an angle. Glass and quartz prisms with an acute refracting angle are commonly occurring examples.

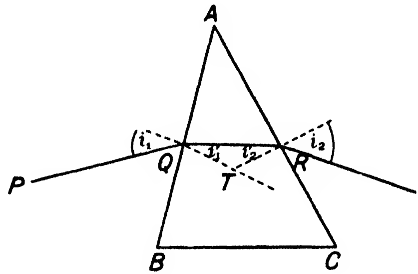

Fig. 3.5

Refraction through a prism (fig. 3.5) thus consists of two successive refractions at the two faces. In the figure the angles of incidence and refraction are denoted by i_1, i_1' and i_2, i_2'.

The deviation of the ray at Q is $(i_1 - i_1')$, and at R $(i_2 - i_2')$. Thus the total deviation is

$$D = i_1 + i_2 - (i_1' + i_2'). \qquad 3.8$$

Since AQT and ART are right angles

$$A = i_1' + i_2', \quad . \quad . \quad . \quad . \quad 3.9$$

and thus

$$D = i_1 + i_2 - A. \quad . \quad . \quad . \quad 3.10$$

If the refractive index of the material of the prism be n,

$$\sin i_1' = \frac{\sin i_1}{n}$$

or

$$i_1' = \arc\sin\left(\frac{\sin i_1}{n}\right).$$

The deviation at the first face is

$$D_1 = i_1 - \arc\sin\left(\frac{\sin i_1}{n}\right).$$

It can be shown that the differential coefficient of D_1 with respect to i_1 is positive, n being greater than unity, i.e. the deviation D_1 increases as i_1 increases.

It follows that the deviation of a ray which passes through the prism is always away from the refracting edge at A. This is true if the incident ray falls on the other side of the normal and if the angle QRA is obtuse.

Minimum Deviation

As the angle of incidence i_1 varies, the deviation D varies with it. A graph of the relation between D and i_1 can be investigated experimentally, when a curve similar to that shown in fig. 3.6 will be obtained. The curve has a minimum corresponding to one of the values of i_1. When the measurements are made carefully, it will be found that at this angle of incidence the ray passes symmetrically through the prism with $i_1 = i_2$ and $i_1' = i_2'$. Thus

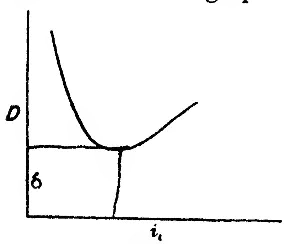

Fig. 3.6

$$i_1' = i_2' = \frac{A}{2}, \text{ and } D = 2i_1 - A,$$

whence

$$n = \frac{\sin i_1}{\sin i_1'} = \frac{\sin \frac{1}{2}(D + A)}{\sin \frac{A}{2}}. \quad . \quad . \quad 3.11$$

This formula is important on account of its use in the determination of refractive indices.

A theoretical proof of the statement that the minimum value of D occurs when $i_1 = i_2$ and $i_1' = i_2'$ can be given. We have

$$\sin i_1 = n \sin i_1', \ \sin i_2 = n \sin i_2'.$$

Add these together

$$\sin i_1 + \sin i_2 = n(\sin i_1' + \sin i_2')$$
$$\sin \tfrac{1}{2}(i_1 + i_2) \cos \tfrac{1}{2}(i_1 - i_2) = n \sin \tfrac{1}{2}(i_1' + i_2') \cos \tfrac{1}{2}(i_1' - i_2')$$

or

$$\sin \tfrac{1}{2}(D + A) \cos \tfrac{1}{2}(i_1 - i_2) = n \sin \tfrac{1}{2}A \cos \tfrac{1}{2}(i_1' - i_2')$$
$$\sin \tfrac{1}{2}(D + A) = n \sin \tfrac{1}{2}A \frac{\cos \tfrac{1}{2}(i_1' - i_2')}{\cos \tfrac{1}{2}(i_1 - i_2)} . \quad . \quad . \quad 3.12$$

If i_1 and i_2 are not equal, let $i_1 > i_2$. We have seen that the deviation $(i_1 - i_1')$ increases with i_1, so that since $i_1 > i_2$, $(i_1 - i_1') > (i_2 - i_2')$, i.e.

$$i_1 - i_2 > i_1' - i_2'.$$

Thus $\cos \tfrac{1}{2}(i_1' - i_2') > \cos \tfrac{1}{2}(i_1 - i_2)$, these deviations being less than a right angle.

The same relation holds if $i_1 < i_2$.

Thus when $i_1 \neq i_2$, $\sin \tfrac{1}{2}(D + A) > n \sin \tfrac{1}{2}A$, but when $i_1 = i_2$,

$$\sin \tfrac{1}{2}(D + A) = n \sin \tfrac{1}{2}A.$$

This relation gives the least value of D, for the inequality shows that D is greater in other cases.

We thus obtain the minimum value of D when $i_1 = i_2$, the ray passing symmetrically through the prism.

Angular Magnification by a Prism

There is a further interesting property at the position of minimum deviation which we can study by considering the angular magnification.

Let two rays PQ and PQ′ be refracted by the prism and be deviated into the directions RS and R′S′ respectively.

Let PQ be incident on AB at an angle i_1, PQ′ at $(i_1 + \delta i_1)$. Then $\angle QPQ' = \delta i_1$.

If the angles of emergence of RS and R′S′ are i_2 and $i_2 + \delta i_2$, the angle SP′S′ is $-\delta i_2$.

The angular magnification means the ratio of these angles SP′S′ to QPQ′ $\left(-\dfrac{\delta i_2}{\delta i_1}\right)$.

From the law of refraction $\sin i_1 = n \sin i_1'$,

$$\cos i_1 \, \delta i_1 = n \cos i_1' \, \delta i_1',$$

and in the same way

$$\cos i_2 \delta i_2 = n \cos i_2' \delta i_2'.$$

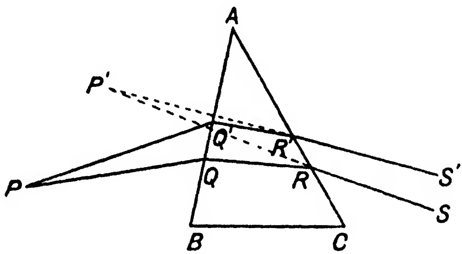

Fig. 3.7

Since

$$i_1' + i_2' = \text{A}, \quad \delta i_1' = -\delta i_2',$$

hence

$$-\frac{\delta i_2}{\delta i_1} = \frac{\cos i_2'}{\cos i_2} \cdot \frac{\cos i_1}{\cos i_1'}. \qquad \text{3.13}$$

At the position of minimum deviation this ratio is unity or the angles QPQ′ and RP′R′ are equal. Thus the prism does not alter the inclination of rays which are incident on the prism in the neighbourhood of the position of minimum deviation.

This may also be seen from the consideration that close to the position of minimum deviation there is very little change in the magnitude of the deviation. Thus if the rays PQ and PQ′ are close to the symmetrical position the deviation will be the same for each, and the emergent rays will be inclined at the same angle as the incident.

This is an important point in the use of prisms in association with lenses as in the spectrometer. Rays from a point will still appear to diverge from a point after passing through a prism, and they can then be focussed by means of a lens. An important application is in the formation of a pure spectrum.

Refraction at Spherical Surfaces

1. *The Concave Refracting Surface*

Let PA (fig. 3.8) denote a spherical boundary separating two media of refractive indices n and n'. Let the source of light be situated on the axis at the point O in the medium

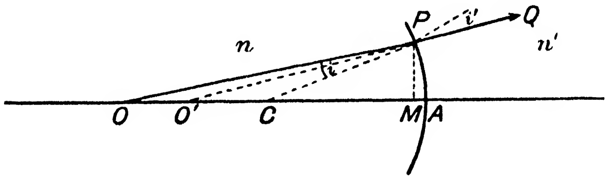

Fig. 3.8

of refractive index n. Let an incident ray be OP and let PQ be the refracted ray which, produced backward, cuts the axis in O'.

The angles of incidence and refraction and the inclinations of the rays to the axis are small, so that the angle in radians and its sine are taken to be equal.

The second law of refraction gives

$$n \sin i = n' \sin i'$$

or

$$ni = n'i', \quad . \quad . \quad . \quad . \quad . \quad 3.14$$

$$\angle \mathrm{CPO}' = i', \ \angle \mathrm{CPO} = i.$$

Thus

$$i = \mathrm{PCM} - \mathrm{POM},$$

$$i' = \mathrm{PCM} - \mathrm{PO'M}.$$

Now

$$\angle PCM = \frac{PM}{MC}, \quad \angle POM = \frac{PM}{MO}, \quad \angle PO'M = \frac{PM}{MO'}.$$

Hence

$$n\left(\frac{PM}{MC} - \frac{PM}{MO}\right) = n'\left(\frac{PM}{MC} - \frac{PM}{MO'}\right)$$

or

$$n\left(\frac{1}{MC} - \frac{1}{MO}\right) = n'\left(\frac{1}{MC} - \frac{1}{MO'}\right). \qquad 3.15$$

The point A is again a self-conjugate point, and for this reason and on account of its position on the surface we shall choose it as origin of co-ordinates.

Let us take the positive direction for measurements of O and of O′ to the left. Thus

$$l = AO, \; l' = AO'.$$

Since the surface is concave towards the light, the radius $r = -AC$. Thus from (3.15), remembering that M and A may be regarded as coincident,

$$n\left(\frac{1}{-r} - \frac{1}{l}\right) = n'\left(\frac{1}{-r} - \frac{1}{l'}\right)$$

or

$$\frac{n'}{l'} - \frac{n}{l} = \frac{n - n'}{r}. \quad . \quad . \quad . \quad 3.16$$

This result is true for all rays making small angles with the axis, so that the surface focusses the rays at O′. When l' is very great we find $l = \frac{nr}{n' - n}$, and when l is very great $l' = \frac{n'r}{n - n'}$.

These two values give the object and image focal lengths respectively, and we have

$$f = \frac{nr}{n' - n}, \; f' = \frac{n'r}{n - n'} \quad . \quad . \quad 3.17$$

Thus (3.16) may be written in the form

$$\frac{f'}{l'} + \frac{f}{l} = 1. \qquad . \quad . \quad . \quad . \quad . \quad 3.18$$

This is of the same form as the formula (2.5) which was obtained for the mirrors.

Change of Co-ordinate System

We shall now take the positive direction of l' to the right, i.e. we write $l = \mathrm{AO}$, $l' = -\mathrm{AO}'$.

(3.15) now becomes

$$n\left(\frac{1}{-r} - \frac{1}{l}\right) = n'\left(\frac{1}{-r} + \frac{1}{l'}\right),$$

or

$$\frac{n'}{l'} + \frac{n}{l} = \frac{n' - n}{r}. \qquad . \quad . \quad . \quad . \quad 3.19$$

Proceeding as before, we find

$$f = \frac{nr}{n' - n}, \quad f' = \frac{n'r}{n' - n}, \qquad . \quad . \quad . \quad 3.20$$

and we obtain once more the form (3.18).

We have now sufficiently illustrated the independence of this form on the co-ordinate system. The reader may verify this independence in all the cases still to be considered, but we shall adopt as the system most convenient for our purpose that in which the object co-ordinate is measured positively towards the left and the image co-ordinate positively towards the right. The reason for this choice will become evident when we come to consider lenses.

Whenever the point A with its self-conjugate character exists, we shall choose this point as origin, or refer our measurements to the focal points. In our diagrams we shall place the object on the left as illustrated in this chapter.

The Focal Points and Focal Distances

In this case the focal points do not coincide, and the focal lengths are unequal.

From (3.20) we see that f and f' are of the same sign, and in the system of co-ordinates employed this means that the focal points lie on opposite sides of A.

If the radius is 10 cm., $n = 1$ and $n' = 1{\cdot}5$, the value of f is -20 cm., i.e. the object focal point lies 20 cm. to the right of A. The value of f' is -30 cm., and the

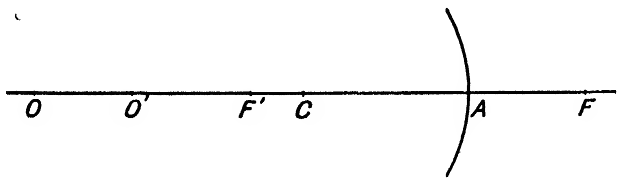

Fig. 3.9

image focal point lies 30 cm. to the left of A. The positions in every case will depend upon the values of the radius and refractive indices. In fig. 3.9 we place F to the right and F′ to the left of A.

The Focal Points as Origins

We shall take the object focal point F as origin for the positions of the object, and the distances FO will be denoted by x. The positions of the image O′ will be measured from the image focal point F′, and the distances denoted by x'.

In the example of the mirrors the simplest co-ordinate system is that in which directions to the right are taken as positive, for we have a single origin for both object and image measurements, whether we measure from A or from F. But in the more general cases where two origins are taken and where they may lie on one side or the other of the refracting surface, there is no advantage of one system over the other. We shall, following the

convention adopted, measure image positions positively to the right.

We have

$$x = \mathrm{FO} = \mathrm{AF} + \mathrm{AO}$$
$$x' = -\mathrm{F'O'} = -\mathrm{AO'} + \mathrm{AF'},$$

reference being made to fig. 3.9.

But

$$f = -\mathrm{AF},\ f' = -\mathrm{AF'},\ \text{and } l = \mathrm{AO},\ l' = -\mathrm{AO'}.$$

Thus

$$x = -f + l,\ x' = l' - f',$$

and substituting $l = x + f$, $l' = x' + f'$ in (3.18), we obtain

$$xx' = ff'. \qquad 3.21$$

This is identical in form with the formula for the mirrors, the difference being in the values of f and f'.

Refraction at a Convex Surface

The procedure is the same as that for the concave surface. Referring to fig. 3.10, we have

$$i = \mathrm{PCM} + \mathrm{POM}$$
$$i' = \mathrm{PCM} + \mathrm{PO'M},$$

and as before (3.14), $ni = n'i'$.

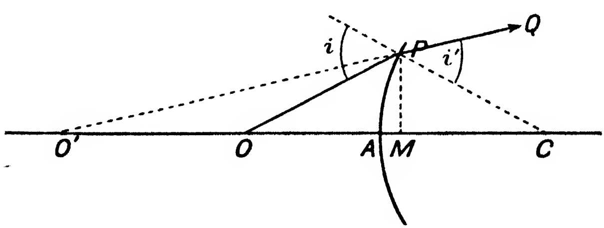

Fig. 3.10

Now

$$\angle\mathrm{PCM} = \frac{\mathrm{PM}}{\mathrm{MC}},\ \angle\mathrm{POM} = \frac{\mathrm{PM}}{\mathrm{MO}},\ \angle\mathrm{PO'M} = \frac{\mathrm{PM}}{\mathrm{MO'}}.$$

Hence

$$n\left(\frac{1}{\mathrm{MC}}+\frac{1}{\mathrm{MO}}\right)=n'\left(\frac{1}{\mathrm{MC}}+\frac{1}{\mathrm{MO'}}\right). \qquad 3.22$$

This should be compared with (3.15).

To the approximation we are considering, MO = AO = l, and MO′ = AO′ − l'. The radius of curvature in this case is r = AC, and MC may be replaced by AC; thus from (3.22)

$$n\left(\frac{1}{r}+\frac{1}{l}\right)=n'\left(\frac{1}{r}-\frac{1}{l'}\right)$$

or

$$\frac{n'}{l'}+\frac{n}{l}=\frac{n'-n}{r}, \qquad 3.23$$

which is identical with 3.19.

Thus the two surfaces may be described by the same formula, the difference between them being accounted for by the difference in sign of the radii of curvature.

We can thus sum up the formulæ for the refraction at spherical surfaces by

$$\frac{f'}{l'}+\frac{f}{l}=1, \qquad 3.18$$

$$f=\frac{nr}{n'-n}, \quad f'=\frac{n'r}{n'-n}, \qquad 3.20$$

$$xx'=ff'. \qquad 3.21$$

of a refracting medium, such as glass, situated in air, we have r_1 positive, r_2 negative and $n > 1$.

Thus both f and f' are positive in the particular convention we have used.

2. *The Biconcave Lens*

In this case the first radius r_1 is negative and the second r_2 is positive. The focal lengths are equal and negative.

3. *The Plano-convex Lens*

If the convex surface is turned towards the light and has a radius of numerical value a, $r_1 = a$ and $r_2 = \infty$. Thus

$$\frac{1}{f} = \frac{n-1}{a}.$$

If the plane surface is towards the light, $r_1 = \infty$, $r_2 = -a$ and $\frac{1}{f} = \frac{n-1}{a}$ as before.

The focal lengths in either case are positive and have the same value.

4. *The Plano-concave Lens*

The focal lengths are negative and equal.

Converging and Diverging Lenses. Notation

Rays directed to a point such as PQ and P′Q′ (fig. 4.4), which are directed to S′, may, on passing through a refracting medium such as a parallel-sided plate of glass, emerge with no alteration in their mutual inclinations. They may, as in passing through a lens, become more convergent or more divergent according to the kind of lens. We use the term " power " in describing this property of optical systems. In these examples the parallel-sided plate has zero power. A convex lens in air, which causes rays to become more convergent after they have passed through it, is of positive power, while a concave

lens, which has the opposite effect, has negative power (fig. 4.4).

In considering the amount of convergence or divergence produced by a lens, it will be simplest to consider rays which are originally parallel to the axis, i.e. which come

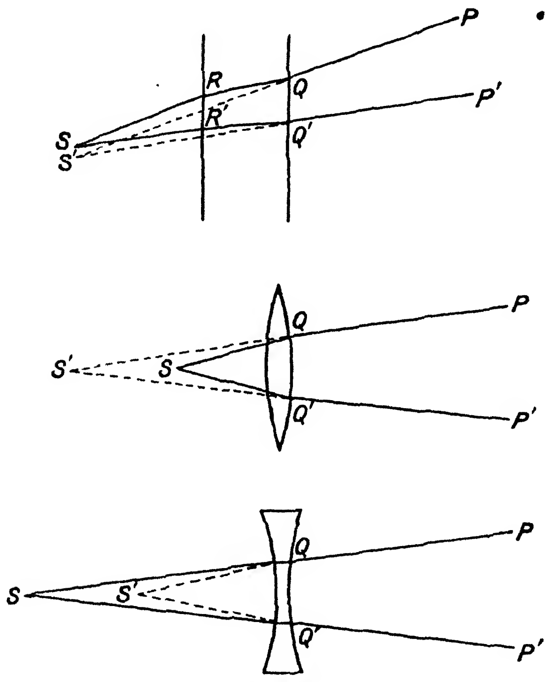

Fig. 4.4

from a distant object. These rays after being refracted pass through the image focal point of the system. If we refer to the case of the convex lens, it is clear that the greater the power of convergence, the shorter the focal length, and for lenses in air the power is measured by the reciprocal of the image focal length $\frac{1}{f'}$. To conform

to an established use this quantity should be positive, i.e. f' should be positive, for a converging system.

We have seen that for the converging lens f' is positive if we measure the positive direction in the image space to the right, and by reference to the method adopted in this book this means that this positive direction is that towards which the light is travelling.

To bring about this agreement in sign we have also to adopt the particular convention with regard to the radii of curvature of surfaces, which we have used here.

We have seen that the object focal length is equal to the image focal length, but that the foci lie on opposite sides of a simple lens. If a lens is turned about so that the front surface takes the place of the back surface, what was before the object focal point becomes the image focal point, and it seems somewhat unsatisfactory to use a notation which distinguishes the focal lengths by different signs. The description of a converging system as positive should mean that both focal lengths are positive. Moreover, the possibility of interchanging object and image is best described in the notation in which the positive direction in the object space is opposite to that in the image space.

For these reasons we shall adopt a double system of co-ordinates for all optical systems, one for the location of the object and another for the location of the image, the positive directions along the axes being opposite in the two cases, and the positive direction in the image system being that of the direction towards which the light is travelling. Directions above and below the axis are measured according to the ordinary rule of co-ordinate geometry.

A system which has been recommended and which many use is a single system for the location of both object and image, in which the positive direction is that towards which the light is travelling. The double system we use here differs from it in a change of sign of the object distance l.

The formulæ (2.5), (2.8), (2.11), (3.20), (4.5) and (4.6) are all given in accordance with the double system.

The least deviation from the methods employed in graphical construction will occur if diagrams are drawn so that the initial direction of the light is from left to right. This will be the case in most examples if the object is placed to the left in the diagram. In this case we shall have the positive direction to the right for the image and the positive direction to the left for the object.

A Further Property of the Fundamental Formula

We have seen in the case of the mirrors that the formula, $\frac{f'}{l'}+\frac{f}{l}=1$, retains its form unchanged if the centre of curvature is taken as the origin instead of the pole of the mirror. It has been stated that this is due to the fact that both these points are self-conjugate.

We shall now see that the origin can be changed in a more general manner without altering the form.

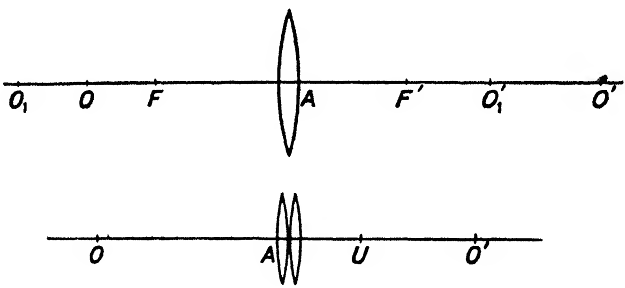

Fig. 4.5

This will be illustrated with special reference to a convex lens (fig. 4.5), but the result is quite general.

Let O_1 and O_1' be a pair of conjugate points with co-ordinates l_1 and l_1', i.e. $l_1 = AO_1$, $l_1' = AO_1'$. Thus

$$\frac{f'}{l_1'}+\frac{f}{l_1}=1. \quad . \quad . \quad . \quad . \quad 4.7$$

Let O and O′ be another pair of conjugate points with co-ordinates l and l'. Thus

$$\frac{f'}{l'} + \frac{f}{l} = 1. \qquad \qquad 4.8$$

Suppose that the origin be changed so that, instead of measuring from A, we measure object distances from O_1 and image distances from O_1'. Denoting the new co-ordinates by L and L′, and measuring to the left and right as before,

$$\begin{gathered} L = -O_1O, \quad L' = O_1'O', \\ L = -(AO_1 - AO) = -l_1 + l, \\ L' = AO' - AO_1' = l' - l_1'. \end{gathered}$$

From (4.8)

$$\frac{f'}{L' + l_1'} + \frac{f}{L + l_1} = 1.$$

Combining this with (4.7), we find

$$L(f' - l_1') + L'(f - l_1) = LL'. \qquad 4.9$$

If we measure the distances of the focal points from the new origins, denoting the distances by F and F′, we have

$$F = -O_1F, \quad F' = -O_1'F'$$

or

$$F = -AO_1 + AF = -l_1 + f$$

and

$$F' = -AO_1' + AF' = -l_1' + f',$$

since in the figure

$$f = +AF, \quad f' = +AF'.$$

Thus (4.9) becomes

$$LF' + L'F = LL'$$

or

$$\frac{F'}{L'} + \frac{F}{L} = 1. \qquad \qquad 4.10$$

We come again to the same form, but F and F′ are the distances of the principal foci from the new origins respectively. We do not describe these distances as the focal lengths, since this term is used for the distances of these points when measured from A.

Thus we see that the fundamental equation has the same form provided that the origins of co-ordinates are themselves a pair of conjugate points. The point A being self-conjugate, and thus a pair of conjugate points which happen to be coincident, also enjoys this property. We shall see later on that A possesses a further optical property which causes it to be chosen as origin of co-ordinates in preference to any arbitrary pair of conjugate points.

The Combination of Two Thin Lenses

Two thin lenses placed together form a combined thin lens. The case is similar to the combination of two refracting surfaces to form a lens which we have considered in this chapter.

Let the lenses have focal lengths f_1 and f_2. Let them be in contact at a point A, and let an object be situated at a point O distant l from A. Let U be the intermediate image situated at a distance L′ from A, and let the final image be at O′ at a distance l' from A (fig. 4.5). For the first lens

$$\frac{1}{\mathrm{L}'} + \frac{1}{l} = \frac{1}{f_1},$$

where $\mathrm{L}' = \mathrm{AU}$.

For the second lens U and O′ are conjugate points, and the object distance is $-\mathrm{AU}$ in the diagram. Thus

$$\frac{1}{l'} + \frac{1}{-\mathrm{AU}} = \frac{1}{f_2}.$$

Combining these equations,

$$\frac{1}{l'} + \frac{1}{l} = \frac{1}{f_1} + \frac{1}{f_2},$$

so that the two lenses together are equivalent to a single thin lens of focal length F, where

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}.$$

Remarks on the Nature of the Formulæ

The fact that the various optical systems considered, mirrors, refracting surfaces and lenses, are all described by the same formulæ (3.19) and (3.21) means that these systems have certain general features in common. These features find their expression in a particular kind of formula.

1. All these systems have the property of producing images of objects, and each object has one and only one image. This means that the equation for the determination of the image co-ordinate l' when the object co-ordinate l is given has only one solution. It is therefore a linear relation.

2. But it is possible to have an image at a finite distance from the pole of the system although the object is at an infinite distance, and, on the other hand, although the image may be at an infinite distance the object may be at a finite distance.

These are the common properties of the optical systems, and the equations describing them must bear these features in their algebraical form.

From $\frac{f'}{l'} + \frac{f}{l} = 1$, we obtain

$$l' = \frac{f'l}{l-f} \quad \text{and} \quad l = \frac{fl'}{l'-f'}. \qquad 4.11$$

These equations give a single value for l' when l is known, and conversely. When $l' = f'$, l is infinitely great, and when $l = f$, l' is infinitely great. Thus they possess properties corresponding to the properties of optical systems. This, of course, must be the case since they

have been derived from an application of the laws of reflection and refraction to these systems.

The equations also show that if l is zero, then l' is zero. This means that when the object is at the origin, the image is also at the origin. In most of the examples treated, this means that when the object is at the pole A, the image is there also. But from (4.10) we see that if L and L′ vanish together, the meaning is that when the object is situated at the origin of the object co-ordinates, the image is at the origin of the image co-ordinates, i.e. the two origins form a pair of conjugate points.

It is interesting to notice that in order to satisfy these general conditions (1) and (2), we could arrive at equations of the type of (4.11) without any reference to the laws of reflection and refraction.

These equations belong to the general type

$$l' = \frac{al + b}{cl + d}. \quad . \quad . \quad . \quad . \quad 4.12$$

CHAPTER V

EXTENDED OBJECTS. FORMULÆ FOR MAGNIFICATION. POWER OF OPTICAL SYSTEMS

We shall now take the case when the object is no longer a point lying on the axis but consists of a body at right angles to the axis. The cases we consider are all symmetrical about the axis, and it is therefore sufficient to consider a plane section such as is illustrated in fig. 5.1 for the case of the concave mirror. The object is the line ON, and we have only to consider the figure rotated about the axis to obtain the complete case we are examining.

In the case of a plane mirror the image of the object may be described as true, i.e. it is an exact picture of the

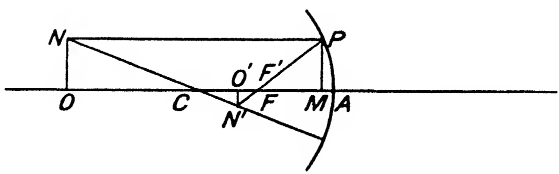

Fig. 5.1

object. There is a difference in that from the point of view of the image the right side has become the left and vice versa, but the straight lines of the original are straight lines in the image, and consequently planes have planes as images. There is no curving of the originally straight parts, and as all the parts of the object are unaltered in size in the image, there is no distortion.

When we pass to more complicated cases of reflection as in curved mirrors, we see that this exact reproduction is not always the case. We have mentioned before that

a large object becomes distorted, and we may reasonably ask if an object such as we have represented by ON in fig. 5.1 gives rise to a straight image such as we have represented by O′N′. Later on we shall have to consider this point from the theoretical aspect, but at this stage we can appeal to experience. We know that small objects placed in the position of ON on the axis do actually give rise to very nearly true images in the case of mirrors. They are sufficiently exact to be regarded as a good approximation for our present purpose. The same is true of the other examples of optical systems such as we have considered.

Our experience of more complicated systems seems at first more convincing, for we know that such instruments as spectacles, opera glasses, telescopes and microscopes, when well made, give faithful reproductions of objects, and some instruments of this kind are used to measure lengths of objects to a high order of accuracy. We are all familiar with the accuracy of reproduction associated with photography. But the problem in these cases is a complicated one, and the instrument maker has to correct for the deviations from the conditions we are now assuming. To these points we shall return in later chapters.

We shall, on these grounds, take it for granted that it is possible to reproduce straight lines in an object as straight lines in an image, and in considering extended objects we shall, in the first instance, consider only such cases, all others being excluded. This means that in the case of the systems we have already considered the extended objects are small.

The Concave Mirror

In Chapter II we have seen how to determine the image O′ of the object O, when this point lies on the axis. Our assumption now is that the image of ON is a straight line and it will clearly pass through O′, but we have to determine the length of O′N′ from a knowledge of that of ON and its position.

Let the point N be at a distance y from the axis, i.e. the co-ordinates of N are (l, y). Let the co-ordinates of N′ be (l', y'), l and l' being measured as before and the positive direction for y being upward, the negative downward. In fig. 5.1, $y = \text{ON}$, $y' = -\text{O'N'}$.

The ratio $\dfrac{y'}{y}$ is called the magnification, because it is the ratio of the size of the image to that of the object.

We can obtain the position of N′ by means of a simple geometrical construction, for a ray from N parallel to the axis will pass through F′(F) after reflection, while a ray through the centre of curvature C will strike the mirror normally and be reflected along CN. The point of intersection of NC and PF gives the point N′, for we assume that the rays from N come to a focus after reflection, and the point where this occurs is localized by two of the rays.

From the figure

$$\frac{\text{O'N'}}{\text{ON}} = \frac{\text{O'N'}}{\text{PM}} = \frac{\text{FO'}}{\text{FA}}, \quad . \quad . \quad . \quad . \quad 5.1$$

A and M being taken as coincident as before.

Although it is again of no particular theoretical importance to choose a particular scheme of co-ordinates for l and l', we shall make use of that which we adopted at the end of the last chapter, leaving it to the reader to verify that the form is unchanged if other systems are adopted.

$$l' = -\text{AO'}, \ \text{AF} = -f'.$$

Thus from (5.1),

$$\frac{-y'}{y} = \frac{-l' + f'}{-f'}$$

or

$$\frac{y'}{y} = \frac{f' - l'}{f'}. \quad . \quad . \quad . \quad . \quad . \quad 5.2$$

We can also write $AF = f$, in the particular case of the concave mirror, and so obtain

$$\frac{y'}{y} = \frac{f + l'}{f},$$

but this is a form which applies to this case only. We prefer (5.2) because, as will be seen, it applies not only to the concave mirror, but to all optical systems, the value of f' being appropriate to the special cases.

We note that if $l' = 0$, which means that the image is at A, and consequently the object is also at A, the ratio is unity. This means that an object placed at A gives rise to an image at A identical with it in size. We can describe A as a point of unit magnification. This property distinguishes A from all other points. It is not merely a self-conjugate point, but one at which the image is of the same size as the object and situated on the same side of the axis. We speak of the plane through A at right angles to the axis as the plane of unit magnification or unit plane. This plane, like the point A itself, is really double—just as A is a point where object and image coincide so the unit plane is strictly the plane in which two unit planes coincide.

By combining (5.2) with the equation

$$\frac{f'}{l'} + \frac{f}{l} = 1,$$

we determine

$$\frac{y'}{y} = \frac{f}{f - l} \qquad \cdots\cdots \quad 5.3$$

and

$$\frac{y'}{y} = -\frac{f}{f'} \cdot \frac{l'}{l} \cdot \qquad \cdots\cdots \quad 5.4$$

We thus have alternative forms for the magnification. All these formulæ are quite general, applying to all optical systems.

The Convex Mirror

The foregoing treatment for concave mirrors is applicable also to convex mirrors. We shall not consider this case separately—the only difference is in the sign of the radius of curvature.

The special values for the case of spherical mirrors are given by inserting the appropriate values of f and f', viz. $f = -\frac{r}{2}$, $f' = \frac{r}{2}$ (2.11). Thus, for example, (5.4) becomes

$$\frac{y'}{y} = \frac{l'}{l}, \qquad \text{5.5}$$

which is most commonly used in applying the formulæ to the case of spherical mirrors.

Extended Objects and Spherical Refracting Surfaces

A comparison of figs. 5.2 and 5.1 shows that the construction to determine the image of an extended object

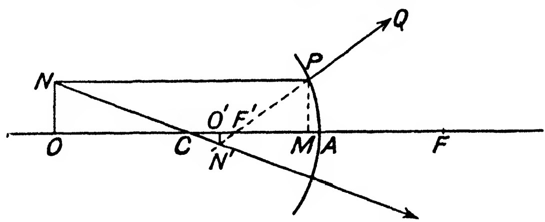

Fig. 5.2

ON by a refracting surface is similar in that of the reflecting surfaces. The two rays NP and NC pass through F′ and C after refraction; the ray NC preserving its direction since it is normal to the surface.

Thus N′ is the image of N, and N′O′, drawn perpendicularly to the axis, is the image of ON.

As before,

$$\frac{O'N'}{ON} = \frac{O'N'}{PM} = \frac{F'O'}{F'A}. \qquad \text{5.6}$$

It is at once evident that we shall arrive at the same formula as before.

$$F'O' = AO' - AF' = -l' + f'.$$

Thus

$$\frac{y'}{y} = \frac{f' - l'}{f'}. \qquad \dots \quad 5{\cdot}7$$

The alternative formulæ are the same as those for the mirror, viz. (5.3) and (5.4).

The value of the magnification applicable to the special case of a spherical refracting surface may be obtained from (5.4) in combination with (3.20). It is found to be

$$\frac{y'}{y} = -\frac{n}{n'} \cdot \frac{l'}{l}. \qquad \dots \quad 5{\cdot}8$$

With an object in air the value of n is unity, and we find

$$\frac{y'}{y} = -\frac{1}{n'} \frac{l'}{l}. \qquad \dots \quad 5{\cdot}9$$

Although the diagram is drawn for the case of a concave surface, the method and formulæ apply to the convex surface also.

Helmholtz's Formula

This important formula gives a relation between the inclinations of conjugate rays to the axis.

Let us first consider it in its application to reflecting surfaces. Let a ray OP (fig. 5.3) be incident upon a reflecting spherical surface either convex or concave, and let PQ denote the conjugate ray. Let a and a' denote the acute angles made by these rays with the axis. The point O lies on the axis of the surface, and ON and O′N′ denote a small object and its image perpendicular to the axis.

A ray NA is reflected along the line through A and N′, making equal angles of incidence and reflection at A.

Then with the usual approximations we have, referring to the figure,

$$PM = OA\alpha = O'A\alpha',$$

$$\frac{ON}{OA} = \frac{O'N'}{O'A},$$

whence

$$ON \cdot \alpha = O'N'\alpha'.$$

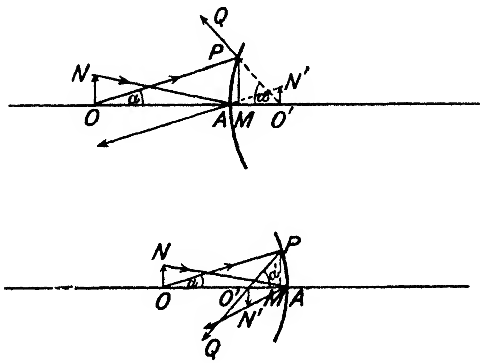

Fig. 5.3

In the case of the convex surface ON and O′N′ are denoted by y and y', and the formula becomes

$$y\alpha = y'\alpha',$$

while in the other case we obtain

$$y\alpha = -y'\alpha'.$$

In order to unite these two formulæ into a single one, it is necessary to adopt some convention with regard to the measurement of angles.

But a convention is already implied in our notation of co-ordinates. For any straight line running so that the x and y of co-ordinates are increasing together makes an angle with the x-axis of which the tangent is positive. Thus in the object space, in accordance with our notation,

the positive inclination is to be associated with a ray passing between the second and fourth circular quadrants, and a ray such as OP in fig. 5.3 has a negative inclination. If α denote the numerical value of the inclination of OP, we must write for the inclination θ the value $-\alpha$, this notation giving not only the magnitude of θ but also its position in the circular quadrants.

Thus we can sum up the convention with regard to angular measurement as follows. The inclination of rays to the axis are to be measured by means of the acute angles between them and the axis and, if the ray passes between the second and fourth quadrants in the object space, the inclination is positive, but it is negative if the ray passes between the other two quadrants. In the same way in the image space a ray passing between the first and third quadrants makes a positive inclination, while the inclination is negative if it passes between the second and fourth quadrants. Thus in the upper diagram of the figure the inclination of OP is given by

$$\theta = -\alpha \quad \text{and} \quad \theta' = -\alpha',$$

and in the lower diagram

$$\theta = -\alpha \quad \text{and} \quad \theta' = +\alpha'.$$

The formulæ thus become

$$y\theta = y'\theta' \qquad . \quad . \quad . \quad . \quad . \quad . \qquad 5.10$$

in both cases.

Let us now examine the case of refraction at a spherical surface, taking the convex and concave surfaces together and referring to the diagram (fig. 5.4) for the notation.

The conjugate rays are OP and PO′ in media of refractive indices n and n' respectively.

AN and AN′ are incident and refracted rays, thus

$$n\frac{\text{ON}}{\text{NA}} = n'\frac{\text{O'N'}}{\text{N'A}}$$

since the angles of incidence and refraction are small. We have also

$$PM = OA\alpha = O'A\alpha',$$

and consequently from these relations

$$nON\alpha = n'O'N'\alpha'.$$

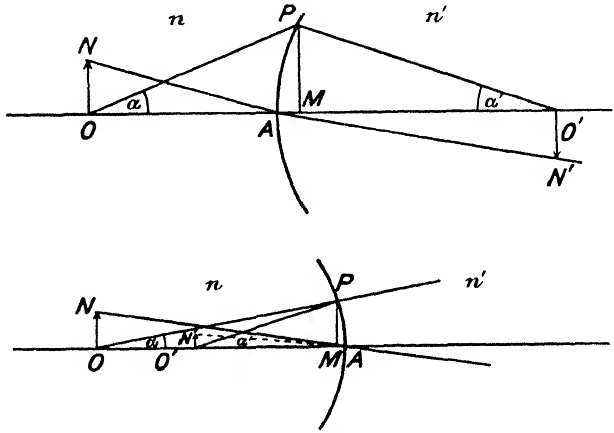

Fig. 5.4

In the case of the convex surface this becomes

$$ny\alpha = -n'y'\alpha',$$

and for the concave surface

$$ny\alpha = n'y'\alpha'.$$

Thus, if we keep the same convention with regard to angles as that adopted in connection with the case of reflection, both cases may be included under the same formula. The case of reflection may also be included in this formula, if for that case we write $\frac{n'}{n} = -1$. The formula may be regarded as showing the relation between

the angular magnification $\frac{\theta'}{\theta}$ for a ray passing through O, and the linear magnification $\frac{y'}{y}$ of an object placed at O.

We shall write θ for the angle of inclination of a ray to the axis, and thus the general formula becomes

$$ny\theta = -n'y'\theta', \quad . \quad . \quad . \quad . \quad . \qquad 5.11$$

In applying it to the upper diagram of fig. 5.4,

$$\theta = -\alpha, \quad \theta' = -\alpha',$$

and to the lower diagram,

$$\theta = -\alpha, \quad \theta' = \alpha'.$$

If this theorem be extended to a succession of surfaces it must be remembered that the angles in the object space are measured with a different sign convention from that of those in the image space. Thus for the refraction at the first surface we have

$$ny\theta = -n'y'\theta',$$

and for the second surface we have a similar relation, which we may write

$$n'y'\theta_1 = -n''y''\theta'',$$

θ' and θ_1 denote the same angle, but θ_1 is now measured in the object space of the second surface. Our convention thus makes $\theta_1 = -\theta'$, and we have

$$n'y'\theta' = n''y''\theta''$$

or

$$ny\theta = -n''y''\theta''.$$

This relation is true, however many surfaces take part in the refraction. Thus in any system, if the conjugate rays make angles θ and θ' with the axis, the relation 5.11 holds.

Thin Lenses

It is now scarcely necessary to consider another case in detail, the method we have adopted being quite general. It consists in localizing N′ by means of two rays, one of which is not deviated by the reflecting or refracting instrument. In the case of the lens this ray passes through the pole, where the medium is practically a thin parallel-sided plate (fig. 5.5).

We have again the same geometric relation

$$\frac{\mathrm{O'N'}}{\mathrm{ON}} = \frac{\mathrm{O'N'}}{\mathrm{PA}} = \frac{\mathrm{F'O'}}{\mathrm{F'A}}.$$

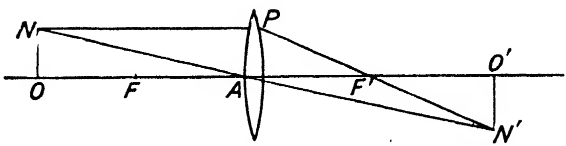

Fig. 5.5

Thus as before,

$$\frac{y'}{y} = \frac{f' - l'}{f'}, \quad . \quad . \quad . \quad . \quad . \quad 5.2$$

and we have in addition (5.3) and (5.4).

In the special case of a thin lens in air $f = f'$, and it follows from (5.4) that

$$\frac{y'}{y} = -\frac{l'}{l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad 5.12$$

The Magnification Formulæ Referred to the Focal Points as Origins

The formulæ expressed in terms of x and x', co-ordinates measured from the focal points F and F′, are sometimes useful. We can without loss of generality refer to fig. 5.5. We have

$$\frac{\mathrm{O'N'}}{\mathrm{ON}} = \frac{\mathrm{F'O'}}{\mathrm{F'A}},$$

where $F'O' = x'$ and $AF' = f'$. Thus

$$\frac{y'}{y} = -\frac{x'}{f'} = -\frac{f}{x} \quad . \quad . \quad . \quad . \quad 5{\cdot}13$$

since

$$xx' = ff'.$$

(5.13) applies, of course, to all the systems.

The Power of Optical Systems

We have referred to the use of the term "power" as applied to optical systems, and have stated that it is related to the image focal length. We shall now give a definition of this term as it is used in optics.

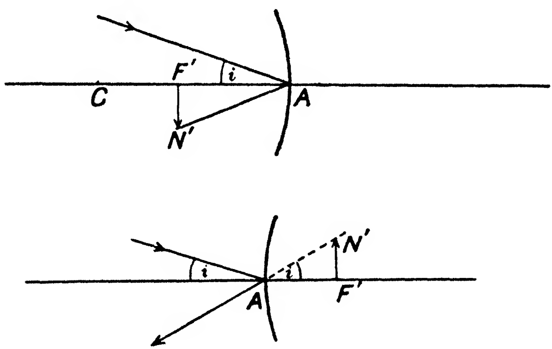

Fig. 5.6

We first distinguish between positive and negative power. An instrument is said to be of positive power if it produces an inverted image of a distant object, and of negative power if it produces an upright image of a distant object.

The numerical value of the power is measured by the small angle subtended by the distant object at a point of the system, divided by the length of its image. Since the object is at a great distance and the system is of finite

dimensions, the result is not altered by taking any point in the system, but it is convenient to choose the pole A for simple systems. In this definition it is assumed that the object is situated in air.

If we apply this definition to the case of a concave mirror (fig. 5.6), we see that the image is inverted and lies with its foot at F(F′). Thus according to the definition the concave mirror is of positive power. The magnitude of the power is

$$\frac{i}{\mathrm{N'F'}} = \frac{\angle \mathrm{N'AF'}}{\mathrm{N'F'}} = \frac{1}{\mathrm{AF'}} = -\frac{1}{f'} = -\frac{2}{r}. \quad 5.14$$

This formula applies in the same way to the convex mirror, but since the image is erect in this case the power

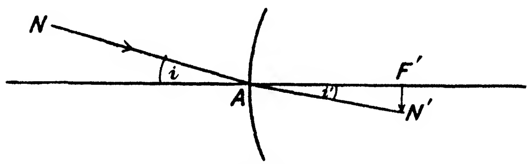

Fig. 5.7

is negative. The formula (5.14) gives the value of the power and the sign agrees with the definition since, according to the convention, the radius of curvature is negative for a concave and positive for the convex surface. This convention was adopted in order to agree with this definition of power. We shall denote the power by the symbol F, and thus for mirrors we have

$$\mathrm{F} = -\frac{2}{r}. \quad \quad 5.15$$

In the case of refraction at a spherical surface, let the angle subtended by the distant object at A be i (fig. 5.7). The ray NA denotes the extreme ray from the object, and the refracted ray AN′ passes through N′, where

N′F′ is the image, situated at the image focal point. The definition of F in this case is

$$F = \frac{i}{N'F'} = \frac{n'i'}{N'F'} = \frac{n'}{AF'} = \frac{n'}{f'}, \quad . \quad 5.16$$

since $i = n'i'$ by the law of refraction, the angles being small and the medium in which the object lies being in air. For this surface f' is positive for a refracting medium denser than air, and again the power agrees in sign with the focal length. On inserting the value of f', we obtain (3.20),

$$F = \frac{n' - 1}{r}. \quad . \quad . \quad . \quad . \quad 5.17$$

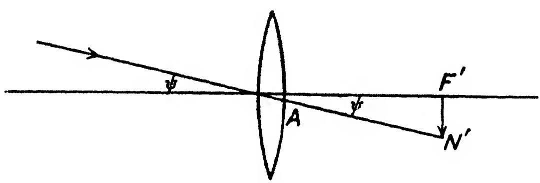

Fig. 5.8

For the convex lens (fig. 5.8) we have

$$F = \frac{\psi}{N'F'} = \frac{\angle N'AF'}{N'F'} = \frac{1}{AF'} = \frac{1}{f'}. \qquad 5.18$$

This is a result to which we have referred before at the end of the last chapter.

Our conventions and scheme of co-ordinates have been chosen so that the sign of F and that of the focal length agree in this case. From (5.18) we see that a lens of positive power has a positive focal length, and one of negative power has negative focal length. It must be borne in mind that this definition of power applies to cases where the rays make small angles with the axis; it is the paraxial value of the power that has been given in the above relations.

The paraxial value is what is usually understood when

we speak of the power of an optical system, but the term is applied in a more general sense to different zones of the system. In these cases the power depends upon the inclination of the rays. The paraxial value, usually described as the power of the mirror, surface or lens, is the value for the central zone which we have been considering.

Unit of Power

It will be noted that the power is measured in terms of the reciprocal of a length, and if the length is expressed in metres the power is said to be measured in dioptres. A dioptre is the unit of power, and from (5.18) we see that, if the focal length of a lens is 1 metre, the power is 1 dioptre. If the lens is convergent, then its power is represented by $+1$D, and if divergent, by -1D.

In the case of a lens of $+20$ cm. focal length, we must express f' in metres, i.e. $\frac{1}{5}$ metre, before inserting the value in (5.18) to obtain the power, which is $+5$D.

Similar remarks apply to other optical instruments.

Examples on Reflection in Curved Mirrors

1. A point is placed on the axis of a concave mirror of radius 40 cm. Find the position of its image when its distance from the mirror is 60 cm.

The formula to be applied in this case is

$$\frac{f'}{l'}+\frac{f}{l}=1.$$

For both types of spherical mirrors

$$f=-\frac{r}{2},\ f'=\frac{r}{2}.$$

In this example we have, according to our convention

$$r=-40 \text{ and consequently } f=20,\ f'=-20.$$

It should be noticed that this means that the object principal focus lies 20 cm. to the left of A, and that the image principal focus lies also 20 cm. to the left of A, both coinciding. In the example the object distance is $l = 60$.

Thus the formula gives

$$\frac{-20}{l'} + \frac{20}{60} = 1,$$

whence

$$l' = -30.$$

In accordance with our convention, this means that the image lies 30 cm. to the left of the point A.

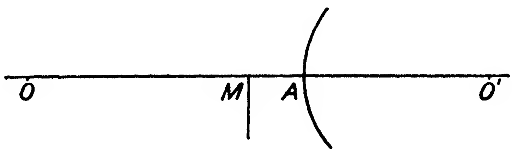

Fig. 5.9

2. A convex mirror and a plane mirror are placed 5 cm. apart with their reflecting surfaces towards an object at a distance of 20 cm. from the plane mirror. The images in the two mirrors coincide. Find the radius of curvature of the convex surface.

Let r be the radius of curvature required.

Then, since for spherical mirrors $f = -\frac{r}{2}$, $f' = \frac{r}{2}$, the formula

$$\frac{f'}{l'} + \frac{f}{l} = 1$$

becomes

$$\frac{1}{l'} - \frac{1}{l} = \frac{2}{r}.$$

O′ and O are the positions of image and object for the

plane mirror as well as for the convex mirror. Thus

$$OM = MO'.$$

But $OM = 20$ cm. and $AM = 5$ cm., thus $AO' = 15$ cm.

In the formula we must write

$$l = 25, \quad l' = 15,$$

$$\frac{1}{15} - \frac{1}{25} = \frac{2}{r},$$

whence

$$r = 75 \text{ cm.}$$

Note that the sign of r is positive, which corresponds to the case of a convex mirror.

Example of Refraction in Spherical Surfaces

3. A small object lies in a cylindrical tank of water, with a thin transparent wall, at a distance of 60 cm. from the curved surface of the tank. If the radius is 50 cm. and the object is viewed by looking at it along a radius, find its apparent position.

This is an example in which the values of f and f' are respectively

$$f = \frac{nr}{n' - n}, \quad f' = \frac{n'r}{n' - n}.$$

The object lies in water, so that $n = \frac{4}{3}$, and the light from it is incident upon a concave surface, thus $r = -50$.

The rays are refracted into air so that $n' = 1$. These data give

$$f = 200, \quad f' = 150.$$

Since, in this case, $l = 60$, the formula for l' is

$$\frac{150}{l'} + \frac{200}{60} = 1,$$

whence

$$l' = -64\tfrac{2}{7}.$$

This means that the image lies at a distance of $64\frac{2}{7}$ cm. from the surface of the tank on the negative side of the image space, i.e. in this case in the water.

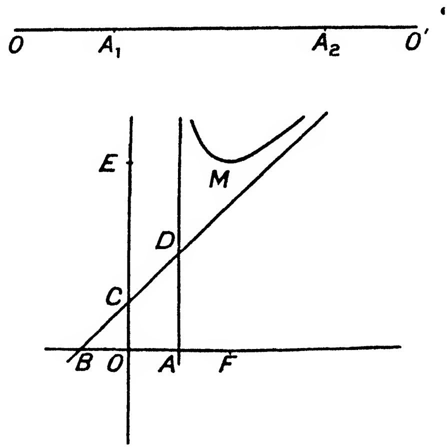

Fig. 5.10

Example on the Simple Lens

4. It is required to throw an image of an object on to a screen by means of a simple convex lens. Show that this is impossible unless the distance between the object and screen is at least four times the focal length, f, of the lens.

In the case of a simple lens in air $f = f'$, and the equation may be written

$$\frac{1}{l'} + \frac{1}{l} = \frac{1}{f}.$$

O denotes the position of the object, O′ that of the screen. Let us suppose that $OO' = a$, i.e. $l + l' = a$. The equation gives

$$\frac{1}{a-l} + \frac{1}{l} = \frac{1}{f},$$

and thus l satisfies the quadratic equation

$$l^2 - al + af = 0.$$

Thus the values of l are

$$l = \frac{a \pm \sqrt{a^2 - 4af}}{2}.$$

These values are real only if

$$4af < a^2$$

or

$$4f < a.$$

This is the answer to the question, but the problem has further points of interest.

When the condition is satisfied there are two real values of l, viz.

$$l_1 = \frac{a + \sqrt{a^2 - 4af}}{2} \text{ and } l_2 = \frac{a - \sqrt{a^2 - 4af}}{2}.$$

The sum of these is a.

Thus an image can be formed with the lens at A_1 and also at A_2; moreover, $OA_1 = l_1$, $O'A_1 = OA_2 = l_2$. This corresponds to the possibility of interchange between the object and image.

The images in the two cases differ in size, as we shall see in the examples on extended objects.

This case offers a useful practical exercise.

The convex lens should be set up and a series of positions of object and image found in order to determine a series of pairs of values of a and l.

These should be plotted on a graph with a as ordinate, y, and l as abscissa, x.

Since

$$l^2 - al + af = 0,$$

the equation in y and x is

$$y(x - f) = x^2.$$

This curve is a hyperbola with asymptotes

$$x - f = 0$$
$$y = x + f.$$

This may be verified by noting that the only value of y which satisfies the curve and the line $x - f = 0$ is an infinite value and, since a line and curve of the second degree intersect in two points, the line in this case does so at infinity, and this means that the line is an asymptote. Similarly, in the case of the other asymptote the only values satisfying it and the curve are infinite values.

When x is less than f the value of y is negative, so that this case cannot be examined by experiment.

If we examine the maximum and minimum values of y, we obtain from $\dfrac{dy}{dx} = 0$,

$$\frac{2x}{(x - f)} - \frac{x^2}{(x - f)^2} = 0,$$

whence

$$x = 0 \quad \text{or} \quad 2f.$$

The value $x = 0$ cannot concern us in the experiment, and from $x = 2f$ we deduce $y = 4f$. This corresponds to the minimum distance between object and screen already discussed.

In the figure, AD is the line $x - f = 0$, BD is $y = x + f$. $OA = OB = OC = f$. $OF = 2f$. $OE = 4f$.

The curve represents the part of the graph which can be investigated by experiment. The minimum is at M with co-ordinates $(2f, 4f)$. It is an interesting exercise to determine f by measuring OA, AF, OB, OC and OE.

Examples on Extended Objects

5. An object 4 cm. long is placed at right angles to the axis of a convex mirror of focal length 10 cm. If the distance between the object and the mirror be 12 cm.,

find the position and size of the image. The values of the focal lengths in this case are

$$f = -10, \quad f' = 10.$$

Thus the image distance, l', is obtained from the equations

$$\frac{10}{l'} - \frac{10}{12} = 1.$$

This gives $l' = 5\frac{5}{11}$ cm., and the image thus lies at this distance behind the mirror.

The formulæ for magnification are

$$\frac{y'}{y} = \frac{f}{f-l} = \frac{f'-l'}{f'} = -\frac{f}{f'} \cdot \frac{l'}{l}.$$

In this case the most convenient is the first, since the data necessary are given in the problem

$$\frac{y'}{4} = \frac{-10}{-10-12} = \frac{10}{22}.$$

Thus $y' = 1\frac{9}{11}$ cm., so that the image is erect and of this length.

6. Let us return to example 4, and suppose that the object is of length y.

When the value of l is l_1, the corresponding value of l' is l_2. Let us suppose that l_1 is the smaller of the two values.

The formula for magnification applicable here is the third of those given in example 5, viz.

$$\frac{y'}{y} = -\frac{f}{f'} \cdot \frac{l'}{l}.$$

In this case $f = f'$, so that we have

$$\frac{y'}{y} = -\frac{l_2}{l_1}.$$

When the value of l is l_2, the corresponding value of l' is l_1, and

$$\frac{y''}{y} = -\frac{l_1}{l_2},$$

where y' is the length of the image in the first case, and $y' > y$ so that the image is a magnified one, while y'', the length in the second case, is less than y and the image is a diminished one. Thus

$$y^2 = y'y''.$$

This formula is of practical use in finding the length of an object from that of the two images which may be thrown on a screen by movement of a convex lens between object and screen. It is often made use of in the experiment on interference fringes formed by means of Fresnel's biprism, and it is important to remember that the application will only succeed if the distance between the object and screen is at least four times the focal length of the lens. In the biprism experiment this means that the distance between the slit and cross wires is subject to this limitation.

CHAPTER VI

GENERAL TREATMENT OF OPTICAL SYSTEMS
COMBINATION OF SYSTEMS
TELESCOPIC SYSTEMS

The systems considered in the preceding chapters may be described as simple optical systems. They all possess a single point A, which is self-conjugate and which has the property that an object placed there gives rise to an image of equal size coincident with the object. In other words, the unit planes pass through A. We have seen that the focal points, which in general are separate, can under certain conditions coincide and thus endow the particular system with geometrical simplicity.

Systems occur in which the unit planes are not coincident and in which the dual character of A becomes evident. We shall have to replace it by two points, which will be denoted by H and H′. Such systems will be distinguished by describing them as complex. They are generally of more complicated structure than those already considered, consisting of two or more lenses or mirrors, but the complexity which appears in our description lies in the duplicity of A, and an apparently simple structure like a thick lens is, according to this definition, to be described as complex.

We shall find that certain general considerations, for which our treatment of simple systems has prepared us, will enable us to obtain formulæ applicable to all complex systems.

In deriving these general formulæ we shall not have to refer to details of the structure of the systems. These details concern us only when we come to deal with a

particular example. The formulæ will be found to be identical with those for the simple systems.

If we refer to the remarks on the nature of the formulæ at the end of Chapter IV, the reason for this becomes evident. All optical systems with which we are familiar, camera lenses, microscopes, telescopes, which are used to form images, have the property that to every object there is one and only one image, and conversely. Some have the property that an infinitely distant object produces an image at a point at a finite distance from the system, and conversely an object at a finite distance may give rise to an image at infinity, and others have the property that when the object is at infinity the image is also at infinity. This latter group has thus infinite focal distances, and we can divide optical systems into two groups according as the focal distance is finite or infinite. The latter group is described as telescopic, and we shall consider it separately. We have not called attention to it hitherto, although we have had a simple example in the plane mirror, for an object infinitely distant from a plane mirror and lying in front of it gives rise to an infinitely distant image behind it. Another example is that of a telescope focussed on a distant object so that the emerging rays are parallel. When a telescope is used to view distant objects and the eye of the observer is at rest, the telescope is acting as a telescopic system, and the use of the term is due to this fact.

We have seen (4.10) that, if a pair of conjugate points is chosen as origin, one for the object space and the other for the image space, the equation which satisfies these conditions is

$$\frac{f'}{l'} + \frac{f}{l} = 1,$$

and that this equation belongs to the more general type

$$l' = \frac{al + b}{cl + d}. \qquad \qquad 6.1$$

If we compare this equation with (4.11), we see that the latter is the special case when

$$b = 0, \quad \frac{a}{c} = f', \quad \frac{d}{c} = -f.$$

The relation for l in terms of l' is of the form

$$l = \frac{a'l' + b'}{c'l' + d'}, \quad . \quad . \quad . \quad . \quad 6.2$$

and again comparing with (4.11) we see that in this case

$$b' = 0, \quad \frac{a'}{c'} = f, \quad \frac{d'}{c'} = -f'.$$

The constants of (6.1) and (6.2) are not independent of one another, for l can be expressed in terms of l' from (6.1), but we are here interested in the general form of the relations. From (6.1) we see that when $l = 0$, $l' \neq 0$, but is equal to $\frac{b}{d}$. If we interpret this in terms of an object and image relation, this means that when the object is at its origin the image is not at its origin. In other words, in this system of measurement the two origins are not conjugate points. When this is the case both b and b' vanish.

Equations (6.1) and (6.2) represent the object and image relation for any origins taken on the axis. It will naturally be most convenient to simplify these relations as much as possible to suit optical systems, and the first simplification is to get rid of the constants b and b' by choosing the origins in the way indicated. Take a pair of conjugate points as origins.

The equations between object and image distances are now

$$l' = \frac{al}{cl + d}, \quad l = \frac{a'l'}{c'l' + d'}, \quad . \quad . \quad 6.3$$

and these can be simplified by dividing throughout by c and c' respectively. We obtain

$$l' = \frac{\frac{a}{c}l}{l + \frac{d}{c}},$$

which we shall write

$$\left.\begin{aligned} l' &= \frac{Al}{l + B} \\ l &= \frac{A'l'}{l' + B'} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad 6.4$$

and similarly

The general arrangement of origins (M and M′) and a pair of conjugate points O and O′ is shown in fig. 6.1.

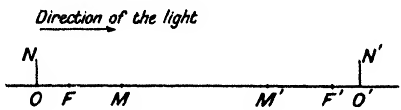

Fig. 6.1

With the direction of the light as shown, the figure agrees with the notation we propose to adopt. It is of no consequence whether we place O on the right or left of the figure, but we must measure l and l' in opposite directions and l' positively with the direction of the light in order to keep to our convention.

Thus $l = \text{MO}$, $l' = \text{M}'\text{O}'$.

When l is infinitely great, we have from (6.4)

$$l' = \frac{A}{1 + \frac{B}{l}}$$

and

$$\frac{1}{l} = 0, \text{ or } l' = A.$$

Let the point F' be at distance A from the origin M'. This point F' is the image focal point of the system, for it is the conjugate of the infinitely distant object.

$M'F'$, the distance of this point from the origin, is not to be called the focal distance, because, as we shall see later, this term is given to the distance of F' from a particular point, which corresponds to the point A of the simple systems. In order to avoid confusion let the distance $M'F'$ be denoted by g', or $A = g'$.

The equation can be written

$$l' = \frac{g'l}{l + B}. \quad . \quad . \quad . \quad . \quad . \quad 6.5$$

From this it follows that

$$l = \frac{Bl'}{g' - l'}. \quad . \quad . \quad . \quad . \quad . \quad 6.6$$

Since this must be identical with (6.4), we obtain the values of A' and B' in this equation in terms of the constants in (6.5).

When the image is at an infinite distance, the object is at the object focal point. Let this be denoted by F.

From (6.6),

$$l = \frac{B}{\dfrac{g'}{l'} - 1},$$

so that when l' is infinite, $l = -B$. Thus $MF = -B$. We shall denote the distance MF by g.

Thus $B = -g$ and (6.6) becomes

$$l = \frac{gl'}{l' - g'}.$$

This may be put into the form

$$\frac{g'}{l'} + \frac{g}{l} = 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad 6.7$$

and this is the same as the equation for the simple systems.

If this method of deviation be examined, it will be seen that we have simply made use of the properties of any optical system, which we described under (1) and (2) at the end of Chapter IV, p. 53.

The difficulty at first is to accept a result which makes no direct reference to the laws of reflection or refraction, nor to the particular media and surfaces which are responsible for deviating the rays in their passage from object to image. It is as a consequence of these laws that optical systems have their particular properties, and we are not really neglecting these laws. These laws produce certain results, and it is from these results that we make our deductions.

Then again, we have not referred to the surfaces, their positions and curvatures. These quantities, however, determine the values of the constants g and g', and, as we shall see, they determine our choice of origin.

Direct Treatment of Two Separated Convergent Lenses

We shall consider the case of two convergent lenses with focal lengths f_1 and f_2, i.e. image and object focal lengths in one case f_1 and in the other f_2.

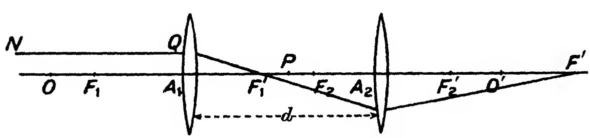

Fig. 6.2

Let us suppose that they are separated by a distance d, and let the relative positions of the foci be as shown. It will be seen that $f_1 + f_2 < d$ in the figure (6.2).

Let O denote a position of the object.

It is a matter of experience that these two lenses produce a single final image O'. But if we follow the process

in detail, we have first to find the image of O in A_1. This can be done by means of the formula

$$\frac{f_1}{l_1'} + \frac{f_1}{l_1} = 1, \qquad . \quad . \quad . \quad . \quad 6.8$$

where $l_1 = A_1O$ and $l_1' = A_1P$, P being the position of the image for refraction by A_1 alone.

P now becomes the object for the second lens, and we denote the distance A_2P by l_2 and A_2O' by l_2'.

We have, for refraction in the second lens,

$$\frac{f_2}{l_2'} + \frac{f_2}{l_2} = 1, \qquad . \quad . \quad . \quad . \quad 6.9$$

and since

$$A_1P + PA_2 = d,$$

$$l_1' + l_2 = d. \qquad . \quad . \quad . \quad 6.10$$

By eliminating l_1' and l_2 from these equations, we can find l_2', which gives the position of O'.

In such a case this procedure is not difficult and not particularly long when actual numbers are the data, but if the system consists of more than two such simple refracting bodies the process is very tedious.

We can also very readily determine the positions of the principal foci, for any ray parallel to the axis, such as NQ, passes after refraction by the first lens through F_1' and is then refracted by the second lens to cut the axis at a point F'. This point F' is the image principal focus for the system consisting of the two lenses taken together. We have now to decide upon the focal length of the system, and this brings us to the question of a suitable pair of origins of co-ordinates. The points A_1 and A_2 lose their significance in a combined system; they are no more than the positions of the lenses.

The whole question is best treated by general considerations, and we can then determine how to treat special cases and how to locate the important points of

the system, such as the principal focal points, as well as to determine fundamental constants of the system.

Extended Objects in Complex Systems

We must now examine the case of extended objects, assuming that the systems we consider are such that plane objects give rise to plane images. We are making the same assumptions as those in Chapter V, p. 56.

As before, let y denote ON and y' denote ON′ (fig. 6.1). The size of the image will depend upon the position and upon the size of the object, i.e. it will depend upon l and y. For every point N there is one and only one point N′, and in the systems we are now considering it is possible for an infinitely distant object to give rise to an image at a finite distance and conversely.

Thus the relation between y' and l and y must be similar in form to (6.1), i.e. the general form of the relation is

$$y' = \frac{al + by + c}{dl + ey + g}. \quad . \quad . \quad . \quad 6.11$$

This form should be compared with the special cases already considered in Chapter V. We do not suggest that the constants in this formula are in any way related to those of (6.1), a, b, c, etc., denote any constants.

When $y = 0$, i.e. when the object is any point O on the axis, we know that the image is a point O′ or $y' = 0$. This enables us to simplify the general formula (6.11). For we have the result that whenever $y = 0$,

$$y' = 0 = \frac{al + c}{dl + g}$$

for all values of l. This can only be the case if a and c vanish.

Thus the formula reduces at once to

$$y' = \frac{by}{dl + ey + g}. \quad . \quad . \quad . \quad 6.12$$

We can simplify this still further by means of a property of optical systems. Consider the ratio

$$\frac{y'}{y} = \frac{b}{dl + ey + g}.$$

This ratio is the magnification, and in the optical systems which we are considering and which reproduce the object faithfully to scale it is impossible for the magnification to depend upon y. If it did it would mean that objects of one length would be magnified to a different extent from those of another length. In a photograph produced by such a system the figures in the picture

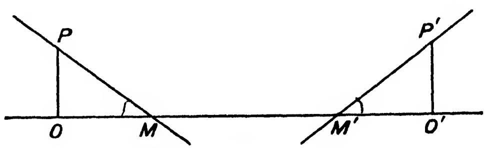

Fig. 6.3

would not preserve the same relative sizes as they had in the group which was photographed. This would mean that the image was distorted, and for the present we exclude cases of distortion. Thus the term ey cannot occur in the cases we are studying, and to bring this about e must be zero. The formula is now

$$y' = \frac{by}{dl + g} = \frac{Cy}{l + D}, \qquad \ldots \quad 6.13$$

where we have divided throughout by d and made use of new letters for the constants.

Comparing this with (6.7) we notice a similarity of form, but the constants differ.

There is a further property which we have not yet considered and which will lead to a further simplification.

Let M and M′ (fig. 6.3) denote the origins as before, and let MP be a ray in the object space through the origin. There will be a corresponding ray after the light has

passed through the system travelling through M′, which may be denoted by M′P′. The origins are conjugate points, so that a ray through M passes through M′ on emergence.

In the figure we denote only parts of the rays in the two spaces. Let P denote a point on the ray in the object space, and let P′ be its conjugate point. This may be more convincing to the reader, if he takes a simple case such as occurs with a mirror or lens in order to avoid the indefinite path through the complex system.

Let us find the relation between tan PMO and tan P′M′O′. Since P and P′ are conjugate points, the line PO may be taken to represent an object and P′O′ the corresponding image. Thus if $y = \mathrm{PO}$, $y' = \mathrm{P'O'}$, and if $\mathrm{MO} = l$, $\mathrm{M'O'} = l'$.

$$\tan \mathrm{PMO} = \frac{\mathrm{PO}}{\mathrm{OM}} = \frac{y}{l} \quad \text{and} \quad \tan \mathrm{P'M'O'} = \frac{y'}{l'}.$$

But from (6.13) and (6.4)

$$\frac{y'}{l'} = \frac{\mathrm{C}y}{\mathrm{A}l} \cdot \frac{l + \mathrm{B}}{l + \mathrm{D}}$$

or

$$\tan \mathrm{P'M'O'} = \frac{\mathrm{C}}{\mathrm{A}} \cdot \frac{l + \mathrm{B}}{l + \mathrm{D}} \cdot \tan \mathrm{PMO}. \qquad 6.14$$

In measuring the tangents of these angles it is of no consequence where we take the points P and P′ on the lines. Any two conjugate points may be taken. Thus the relation (6.14) cannot depend upon l. Thus $\frac{l + \mathrm{B}}{l + \mathrm{D}}$ is independent of l, and this is only true for all values of l if $\mathrm{B} = \mathrm{D}$.

Thus the denominator of the equation for l' (6.4) is the same as that of the equation for y' (6.13).

We can thus write

$$l' = \frac{\mathrm{A}l}{l + \mathrm{B}}, \quad y' = \frac{\mathrm{C}y}{l + \mathrm{B}}, \quad . \quad . \quad 6.15$$

or by (6.7),

$$l' = \frac{g'l}{l-g}, \quad y' = \frac{Cy}{l-g}. \quad . \quad . \quad 6.16$$

We have seen that definite meanings can be given to g and g', these quantities denoting the distances of the principal foci from the origins respectively.

The constant C has not yet been interpreted as a quantity related to the optical system.

It is convenient to speak of the focal planes as the planes through the focal points at right angles to the axis. Points in these planes have conjugate points at infinity. The equation $l - g = 0$ denotes the object focal plane, and it is clear that for points in this plane l' is infinitely great. A similar remark may be made with regard to the image focal plane $l' - g' = 0$.

A pencil of rays from a point on the object focal plane will emerge as a bundle of parallel rays after passing through the system.

Magnification

The second equation of (6.16) gives the linear magnification of the system, which we can write

$$m = \frac{y'}{y} = \frac{Cy}{l-g}. \quad . \quad . \quad . \quad 6.17$$

The Unit Planes

When the magnification is unity, y and y' are of the same sign and equal in magnitude. O′N′ = ON (fig. 6.1).

Fig. 6.4

From (6.17) we can find the value of l for which this relation holds. The equation gives a single value $l = g + C$. Let this point be denoted by H (fig. 6.4).

Thus MH $= g + C$.

The actual relation between the positions of the points on the axis will depend on the particular optical system. The length M′H′ can be obtained from the value of l by means of the equation

$$l' = \frac{g'l}{l - g} = \frac{g'(g + C)}{C}.$$

This is the value of M′H′.

Thus H and H′ are two conjugate points with the property that an object placed at right angles to the axis at H gives rise to an image at H′ at right angles to the axis and equal to it in size. The planes through these points normal to the axis are called the unit planes, and H and H′ are called the principal points.

The Principal Points as Origins

These points are usually taken as origins, H for the object space and H′ for the image space. The point A of the simple systems is thus a double point, the simple systems having the property that H and H′ coincide. We shall see that this is not the case for more complicated cases.

The focal distances are measured from these points. HF is the object focal distance f, and H′F′ the image focal distance f'.

Now

$$\begin{aligned} \text{MF} = g \quad \text{and} \quad \text{HF} &= \text{MF} - \text{MH} \\ &= g - (g + \text{C}) \quad \text{or} \quad \text{C} = -f. \end{aligned}$$

Also

$$\begin{aligned} f' = \text{H}'\text{F}' &= \text{M}'\text{F}' - \text{M}'\text{H}' \\ &= g' - \frac{g'(g + \text{C})}{\text{C}} = -\frac{gg'}{\text{C}} = \frac{gg'}{f}. \end{aligned}$$

Let the distances of O and O′ be measured from H and H′, and for the moment let HO be denoted by L and

H'O' by L'. Then MO = MH + HO or $l = g + \mathrm{C} + \mathrm{L}$, and similarly $l' = \frac{g'(g + \mathrm{C})}{\mathrm{C}} + \mathrm{L}'$,

i.e.

$$l = g - f + \mathrm{L}, \quad l' = g' - f' + \mathrm{L}'.$$

Thus the equations (6.16) become

$$\frac{f'}{\mathrm{L}'} + \frac{f}{\mathrm{L}} = 1, \quad y' = \frac{fy}{f - \mathrm{L}} = \frac{f' - \mathrm{L}'}{f'} y. \qquad 6.16a$$

We shall, however, understand by l and l' the distances measured from the principal points, and thus write the fundamental equation, as before, in the form

$$\frac{f'}{l'} + \frac{f}{l} = 1, \quad . \quad . \quad . \quad . \quad . \qquad 6.18$$

$$y' = \frac{fy}{f - l} = \frac{f' - l'}{f'} y = -\frac{f}{f'}\frac{l'}{l} y. \quad . \qquad 6.19$$

The former is the fundamental equation for the conjugate points we have already used in the simple systems, and (6.19) should be compared with (5.2) and (5.3). The form of the relations is identical for simple and complex systems; the difference is that in the latter we have two separate origins instead of a single one.

The Magnification Referred to the Principal Foci

If the point O be referred to F as the origin and O' to F', denoting the distances by x and x', we obtain at once as with the simple systems

$$xx' = ff', \quad . \quad . \quad . \quad . \quad . \qquad 6.20$$

$$x = \mathrm{FO} = \mathrm{HO} - \mathrm{HF} = l - f, \text{ and similarly } x' = l' - f';$$

thus from (6.19),

$$\frac{y'}{y} = -\frac{f}{x} = -\frac{x'}{f'}. \quad . \quad . \quad . \quad . \qquad 6.21$$

An Important Property of the Unit Planes

Let a ray OP cut the unit plane of the object space in P and let O′P′ be the ray conjugate to the ray OP (fig. 6.5). The points on the lines may be arranged in pairs of conjugate points, for example, O, O′ and P, P′ are pairs of such points. Now an object at PH gives rise to an equal image in the image unit plane, so that any ray such as OP which passes through the upper end of the object PH must give rise to a ray P′O′ which passes through the upper end of the image P′H′. In other words, since PH = P′H′, a ray directed to a point P of the object unit plane emerges from a point P′ of the image unit plane which

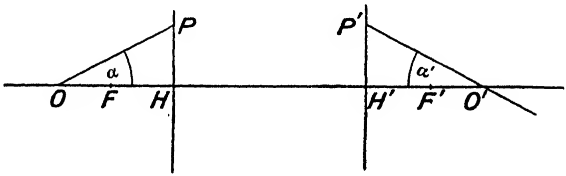

Fig. 6.5

lies on the same side of the axis and is equally distant from it. This is a most important property of the unit planes and is much used in the construction of images.

Focal Lengths Equal when Media on both sides of the System are the same

In the case of a thin lens in air we have seen that both focal lengths have the same value. This is an example of a general property of optical systems, and we shall see that any optical system with the same medium on both sides of it has equal object and image focal distances.

If the system be represented as consisting of a series of refracting spherical surfaces (fig. 6.6), it follows from the theorem of Helmholtz (5.11) that $ny\theta$ remains

unchanged numerically as we pass from medium to medium. Thus

$$ny\theta = -n'y'\theta',$$

the notation being that indicated in the figure, and α and α' being the inclinations of two conjugate rays. The convention for the measurement requires us to give α and

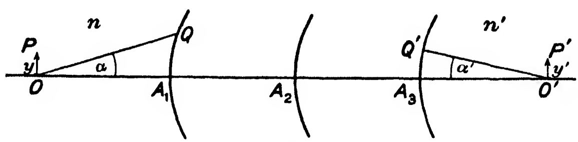

Fig. 6.6

α' each a negative sign. A special case to which this may be applied is that in which the object and image lie in the unit planes.

Consider the case represented in fig. 6.7, which illustrates the case when both focal lengths are positive.

The conjugate to the ray PH is H′P′, and the relative positions are as shown. Now H can be regarded as the

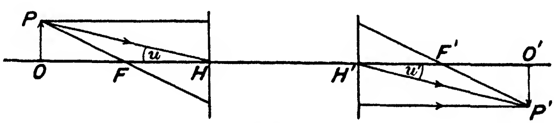

Fig. 6.7

foot of any object of height Y, and H′ the foot of its image, equal to it in size. The angles PHO and P′H′O′ correspond to θ and θ', but u is positive and u' is negative.

Thus

$$nYu = +n'Yu'$$

or

$$nu = +\,n'u',$$

$$n\,\frac{\text{OP}}{\text{OH}} = +n'\,\frac{\text{O}'\text{P}'}{\text{O}'\text{H}'},$$

$$n\frac{y}{l} = -n'\,\frac{y'}{l'}. \qquad \ldots \quad 6.22$$

If this be combined with (6.19), we have

$$\frac{n'}{f'} = \frac{n}{f}, \quad . \quad . \quad . \quad . \quad . \quad 6.23$$

and thus if $n = n'$, $f = f'$.

If this result be applied to (6.19), we obtain for the formula for magnification

$$\frac{y'}{y} = -\frac{n}{n'}\frac{l'}{l}. \quad . \quad . \quad . \quad . \quad . \quad 6.24$$

Angular Magnification

In fig. 6.7 let a ray OP cut the unit plane in P and let the inclination of the ray be α, as shown in the figure. Let the conjugate ray emerge along P′O′, P′ lying in the

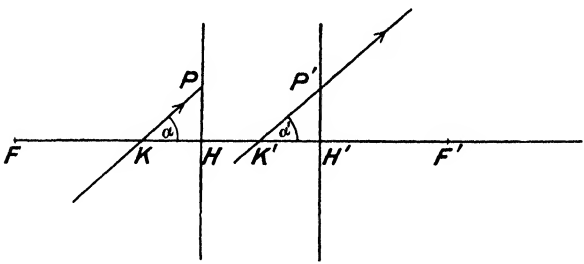

Fig. 6.8

image unit plane and PH and P′H′ being consequently equal. Let this ray be inclined as shown at α' to the axis. Thus in the general notation we have adopted,

$$\theta = -\alpha, \quad \theta' = -\alpha'.$$

We shall measure the angular magnification for a ray and its conjugate by the ratio $\frac{\tan\theta'}{\tan\theta}$, which becomes $\frac{\theta'}{\theta}$ for small angles.

But

$$\text{PH} = \text{OH} \tan \alpha$$

It follows from the geometry of similar figures that:

$$\frac{OF'}{OP'} + \frac{OF}{OP} = 1,$$

thus if $OP = l$, $OP' = l'$, the distance conjugate to l.

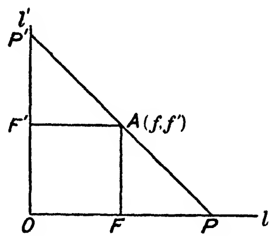

Fig. 6.10

The diagram also explains a familiar method of recording results in order to find the focal lengths of systems.

If pairs of points such as P and P′ are joined, the lines such as PP′ will all pass through A, and hence OF and OF′ may be determined, and in the case of mirrors and lenses in air OF = OF′.

2. *Points not on the Axis*

In this case a different construction is required (fig. 6.11). Let M denote the given point. The construction

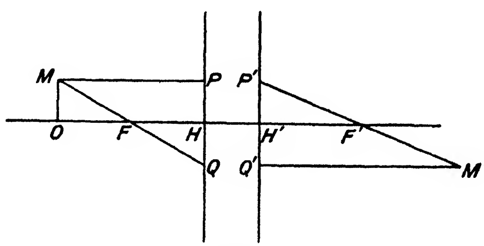

Fig. 6.11

to find M′ is explained by reference to the figure, which explains itself.

The construction of images of extended objects may be obtained by determining the conjugate points of the points bounding the object. In simple cases such as the determination of the image of OM, it is sufficient to find M′ and drop a perpendicular M′O′ to the axis.

3. *Conjugates to a set of Parallel Rays*

Let two parallel rays cut the object principal plane in P′ and P″ and let their inclination to the axis be α (fig. 6.9).

We know that these rays after refraction through the system will meet at a point Q on the image principal plane. To determine the distance y' of Q from F′, draw a ray in the object space through F parallel to the rays. If this cut the object principal plane at P, we know that the ray conjugate to FP is parallel to the axis and that it also passes through Q. Thus Q must be at the same distance from F′ as P is from H. Now

$$PH = f \tan\alpha, \text{ thus } y' = f \tan\alpha. \quad . \quad 6.29$$

The Combination of Optical Systems

Optical systems are made up of reflecting and refracting surfaces which may be combined in a variety of ways. One of the commonest ways of building up an optical system is by means of a number of lenses arranged in contact or separated. In investigating such a system we may consider each element separately and, beginning with a particular ray, trace it step by step through the system until it emerges. This is a long and difficult procedure, and the general treatment is more convenient. The constants of the system must depend on those of the individual parts, and a general consideration of the combination of systems enables special cases to be treated without undue complexity.

The problem is to find the cardinal points of a compound system from a knowledge of those of its individual parts.

Let us consider two systems, the first with principal focal points at F_1 and F_1' and the principal points at H_1 and H_1', the second with the corresponding points located at F_2, F_2' and H_2, H_2'. These points are represented in fig. 6.12, and the distance $F_1'F_2$ is known as the separation of the systems. It is denoted by Δ, and is described as

positive when the two systems are, so to speak, clear of one another or F_2 lies to the right of F_1', as drawn in the figure.

We require to find the principal foci and principal points F, F′ and H, H′ for the combined system.

The image focal point may be located by considering an incident ray AP_1, parallel to the axis. After passing through the two systems it will cut the axis in F′, the image focal point of the combination. The tracing of the ray can be followed from the diagram. It will leave the image principal plane of the first system at P_1'

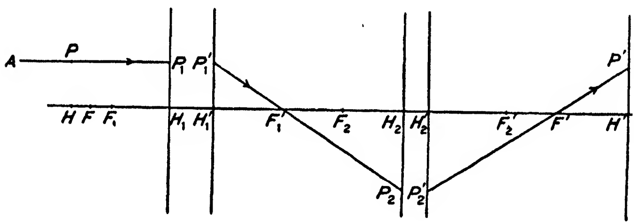

Fig. 6.12

($H_1P_1 = H_1'P_1'$) and pass through F_1', cutting the unit plane of the second system at P_2 and leaving at P_2'.

We do not know where the object principal plane of the combined system is situated, but it will be intersected by the ray AP_1 at some point P, where $PH = P_1H_1$, H being the principal point. The ray emerging will cut the object principal plane at P′, where $P'H' = PH = P_1H_1$, H′ being the image principal point. Thus if we prolong AP_1 to cut the emergent ray $P_2'F'$ in P′, then P′ must lie on the image principal plane of the combination, and on drawing P′H′ perpendicularly to the axis, H′ is located. It remains to determine the length H′F′, and thus the image focal length is given by $f' = -H'F'$, referring to the figure.

F_1' and F′ are conjugate points of the second optical system, F_1' being the object point and F′ the image point.

Thus

$$\frac{f_2'}{H_2'F'} + \frac{f_2}{H_2F_1'} = 1. \quad . \quad . \quad . \quad 6.30$$

But $H_2F_1' = H_2F_2 + F_2F_1' = f_2 + \Delta$, whence

$$H_2'F' = \frac{f_2 f_2'}{\Delta} + f_2' \quad . \quad . \quad . \quad 6.31$$

and

$$F_2'F' = \frac{f_2 f_2'}{\Delta}. \quad . \quad . \quad . \quad . \quad 6.32$$

Either (6.31) or (6.32) serve to locate F', since the second system is known, and hence the points H_2' and F_2' are known.

In order to find the value of f' we can make use of the tangent relation (6.25). In applying this to the present case, we have to be careful to apply the equation correctly.

The angle θ is measured by $\angle P_1'F_1'H_1'$ and θ' by $\angle P'F'H'$ when the formula is applied to the second system. The conjugate points are here F_1' and F', so that

$$\mathcal{U} = H_2F_1', \quad \mathcal{V} = H_2'F'.$$

Thus

$$H_2F_1' \tan P_1'F_1'H_1' = H_2'F' \tan P'F'H'.$$

In addition we have

$$P'H' = P_1'H_1'$$

or

$$H'F' \tan P'F'H' = H_1'F_1' \tan P_1'F_1'H_1'.$$

Thus

$$\frac{H'F'}{H_2'F'} = \frac{H_1'F_1'}{H_2F_1'}$$

or

$$\frac{-f'}{H_2'F'} = \frac{f_1'}{f_2 + \Delta}.$$

Equivalent Lens

If the lenses are separated it is no longer possible to replace the combination by a single thin lens exactly equivalent to it, in the sense that images of all objects are identical in position and size in the two cases. An important difference is obvious at once; the combination has two separate principal points, while the single lens has coincident principal points.

If a lens be constructed of the same focal length as that of the combination, given by (6.44), it is possible to produce an image of an object of the same size both by the system and by the single lens. For if we refer to (6.19) we see that the magnification is given by the focal length and either by the object or image distance.

Thus if the single lens be placed at the object principal point of the system, l is the same for both, and the magnification $\frac{f}{f-l}$ is the same for both systems.

But the image is not in the same place in the two cases, for while the image distance l' has the same value, being given by

$$\frac{f}{l'}+\frac{f}{l}=1$$

($f'=f$ in both cases), in one case l' is measured from the simple lens, in the other it is measured from the image principal point (H′).

An image of the same size and in the same position could be produced by placing the simple lens at H′, but the object would have to be moved so that its distance l from the simple lens was equal to its distance from the principal point H of the combination.

The particular lens which can in this restricted sense be made to replace the combination is called the equivalent lens.

Telescopic Systems

We have seen that in combining simple lenses a case occurs in which the focal length is infinitely great. This property is due to the fact that the image principal focus of the first lens coincides with the object principal focus of the second lens, i.e., referring to fig. 6.16, F_1' coincides with F_2. Thus parallel rays incident upon the first lens come to a focus at the principal focus of the second lens, and so emerge from that lens as parallel rays. This must always occur in the combination of two systems when F_1' and F_2 coincide.

A simpler example of a telescopic system is the plane mirror in which the object image relation is $l' = l$.

Let us refer again to the properties of optical systems (p. 53). The second of these must be replaced by: Objects at infinity give rise to images at infinity. Thus the relation between l and l' is of the simple linear form

$$l' = al + e. \qquad \ldots \qquad 6.48$$

By choosing a pair of conjugate points as origins this simplifies to

$$l' = al. \qquad \ldots \qquad 6.49$$

The same argument as that which led to the relation between the y-co-ordinates (p. 86) leads to a relation similar to (6.13), viz.

$$y' = \frac{cy}{dl + g}. \qquad \ldots \qquad 6.50$$

Let two conjugate points B and B′ be taken as origins (fig. 6.17) and let BP and B′P′ be two conjugate rays. We have

$$\tan \mathrm{PBM} = \frac{\mathrm{PM}}{\mathrm{MB}}, \quad \tan \mathrm{P'B'M'} = \frac{\mathrm{P'M'}}{\mathrm{M'B'}}.$$

Choose P′ conjugate to P so that $\mathrm{PM} = y$ and $\mathrm{P'M'} = y'$, and satisfy the relation (6.50). Then

$$\tan P'B'M' = \frac{y'}{y} \cdot \frac{l}{l'} \tan PBM = \frac{c}{a(dl + g)} \tan PBM.$$

This relation between the tangents is not dependent upon which pair of conjugate points (P, P′) is chosen, so that the relation cannot contain l, which specifies P. Thus d must be zero, and changing the constants (6.50) becomes $y' = by$.

The two equations for telescopic systems are thus

$$\left.\begin{array}{l} l' = al \\ y' = by \end{array}\right\} . \qquad 6.51$$

This simple form holds for any pair of origins provided they are conjugate points.

Fig. 6.17

The linear magnification, $\frac{y'}{y} = b$, is a constant, independent of the conjugate distances. Thus unit planes do not exist in general as in the other systems. In the special case when b is a unity, all conjugate planes are unit planes.

The angular magnification,

$$\frac{\tan\theta'}{\tan\theta} = \frac{y'}{l'} \cdot \frac{l}{y} = \frac{b}{a}, \qquad 6.52$$

is also constant, where θ and θ' are subject to the same sign convention as before. Thus

$$\theta = \angle PBM, \ \theta' = \angle P'B'M'.$$

The product of the linear magnification for an object placed at a particular point and the angular magnification

for a ray through that point is clearly a constant. We have

$$\frac{y'}{y} \cdot \frac{\tan\theta'}{\tan\theta} = \frac{b^2}{a}. \quad \ldots \ldots \quad 6.53$$

In the systems with finite focal distances, we find from (6.19) and (6.25) that this product is given by

$$\frac{y'}{y} \cdot \frac{\tan\theta'}{\tan\theta} = \frac{f}{f-l} \cdot \frac{l}{l'},$$

and by using the fundamental relation (6.18) we find

$$\frac{y'}{y} \cdot \frac{\tan\theta'}{\tan\theta} = -\frac{f}{f'}. \quad \ldots \quad 6.54$$

In telescopic systems f and f' are infinitely great, but from (6.53) and (6.54) we see that, since $\frac{b^2}{a}$ is finite, these focal lengths approach infinity in a definite ratio. In cases where the medium is the same on both sides, this ratio is unity and hence $\frac{b^2}{a} = -1$.

If we consider the example of a combination of two lenses with the distance apart d gradually approaching $(f_1 + f_2)$, f and f' are equal and gradually approach infinity in the ratio 1 : 1. In this case,

$$\frac{y'}{y} \frac{\tan\theta'}{\tan\theta} = -1,$$

or

$$\frac{y'}{y} = -\frac{\tan\theta}{\tan\theta'}. \quad \ldots \ldots \quad 6.55$$

This is the relation between the lateral and angular magnifications for a telescope.

The Combination of Two Systems to give a Telescopic System

If two systems, which are themselves not telescopic, are such that the focal points F_1' and F_2 coincide, they form a telescopic system. For a ray parallel to the axis incident upon the combination emerges as a ray parallel to the axis (fig. 6.18). From (6.33) and (6.34) the ratio of the focal lengths is

$$\frac{f}{f'} = \frac{f_1 f_2}{f_1' f_2'}. \quad . \quad . \quad . \quad . \quad . \qquad 6.56$$

This is the ratio in which the focal lengths f and f' of the telescopic system approach infinity.

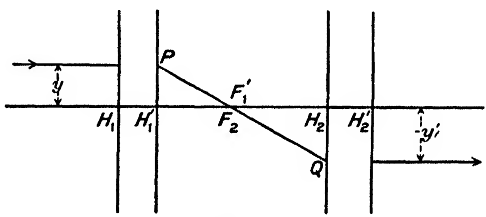

Fig. 6.18

From the figure

$$\frac{y'}{y} = -\frac{QH_2}{PH_1'} = -\frac{H_2F_2}{H_1'F_1'} = -\frac{f_2}{f_1'}. \qquad 6.57$$

We have seen, (6.53) and (6.54), that the ratio $\frac{f}{f'}$ is $-\frac{b^2}{a}$ and $\frac{y'}{y} = b$, so that the constants a and b can be determined for the combination. We have

$$\frac{b^2}{a} = -\frac{f_1 f_2}{f_1' f_2'}, \quad b = -\frac{f_2}{f_1'}, \quad . \quad . \qquad 6.58$$

so that

$$a = -\frac{f_2 f_2'}{f_1 f_1'}. \quad . \quad . \quad . \quad . \qquad 6.59$$

Thus the linear magnification is numerically equal to the ratio of the focal lengths of the separate system

$$\frac{y'}{y} = -\frac{f_2}{f_1'}. \quad . \quad . \quad . \quad . \quad . \quad 6.60$$

The angular magnification is important in this case, and the value of the telescope as an optical instrument lies in the fact that it increases the angle which an object subtends at the eye. The object is made to appear closer.

If we consider an object ray inclined at an angle θ to the axis and with its conjugate ray inclined at θ', we have by (6.52)

$$\frac{\tan\theta'}{\tan\theta} = \frac{b}{a} = \frac{f_1}{f_2'}. \quad . \quad . \quad . \quad . \quad . \quad 6.61$$

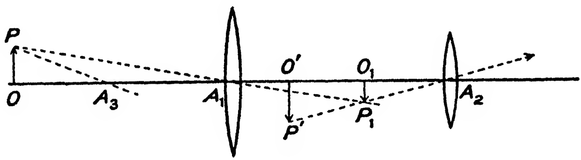

Fig. 6.19

This ratio is called the magnifying power of the telescope, and in the case usually considered with $f_2 = f_2'$, the ratio may be described as the ratio of the focal lengths of the separate systems, i.e. object glass and eye-piece.

The more usual definition of it is the ratio of the angles subtended by the image and the object at the eye.

Let the first system be represented by a lens A_1 and the second by a lens A_2. These are the object glass and eye-piece respectively (fig. 6.19). Let OP denote the position of a distant object and O′P′ that of the final image formed. Since the eye is placed near the eye lens and the object is at a considerable distance away, the ratio of these angles is very approximately equal to $\frac{O'A_2P'}{OA_1P}$.

The ratio of the length of the image to that of the

object is given by (6.57), and it is clear, since $f_1' > f_2$ in telescopes, that there is a diminution in size. But the image is nearer the eye, so that the angle subtended by it may be greater than that subtended by the object.

Let O_1P_1 be the image formed by the object glass, so that PA_1P_1 is a straight line, and let this image be viewed through the eye-piece, O′P′ being the final image.

In the treatment just given y' refers to O′P′ and y to OP; the intermediate image O_1P_1 does not come into consideration.

Since all the angles are small, we can replace them by their tangents,

$$\angle PA_1O = \frac{OP}{OA_1}, \quad \angle P'A_2O' = \frac{O'P'}{O'A_2}.$$

Thus the angular magnification is, according to the usual definition

$$\frac{P'A_2O'}{PA_1O} = \frac{O'P'}{OP} \cdot \frac{OA_1}{O'A_2}. \qquad . \quad . \quad 6.62$$

The first lens produces a magnification of magnitude $-\dfrac{O_1A_1}{OA_1}$ and the second lens of magnitude $+\dfrac{O'A_2}{O_1A_2}$, paying attention to the fact that in the second case $l' = -O'A_2$.

Thus the total magnification

$$\frac{-O'P'}{OP} = -\frac{O_1A_1}{OA_1} \cdot \frac{O'A_2}{O_1A_2}$$

or

$$\frac{O'P'}{OP} = \frac{O_1A_1}{OA_1} \cdot \frac{O'A_2}{O_1A_2}.$$

From (6.62),

$$\frac{P'A_2O'}{PA_1O} = \frac{O_1A_1}{O_1A_2}. \qquad . \quad . \quad . \quad 6.63$$

Since the object is at a great distance from the object glass, $O_1A_1 = f_1$, and on viewing the final image with the

emergent rays parallel, $O_1A_2 = f_2$. This is the case when a normal eye with muscles relaxed is applied to the eye-piece; the eye is then in the condition in which it views distant objects. Thus the right-hand side of (6.63) becomes $\dfrac{f_1}{f_2}$ and has the same numerical value as the angular magnification (6.61). This result also follows from the fact that the ray conjugate to $P'A_2$ passes from P to a point on the axis somewhere in the neighbourhood of the system, e.g. at A_3, where A_3 and A_2 are conjugate points for the telescope. Now $\angle PA_3O$ and $\angle PA_1O$ may be regarded as equal since OP is a distant object. Thus the equation (6.61) applies directly to the rays PA_3 and $P'A_2$. For small angles (6.61) gives the same numerical value, for the ratio $\dfrac{PA_3O}{P'A_2O'}$ or $\dfrac{PA_1O}{P'A_2O'}$ is equal to $\dfrac{f_1}{f_2}$, since $f_2' = f_2$ in this case, where $\theta = \angle PA_3O$ and $\theta' = \angle P'A_2O'$.

The Combination of Two Optical Systems one of which is Telescopic

In defining an optical system which is not telescopic the cardinal points are required, but in a telescopic system all that is required are the constants a and b, occurring in the equations

$$l' = al, \quad y' = by.$$

We can choose as origins any pair of conjugate points.

Suppose that the non-telescopic system has principal points at H_1 and H_1'. We shall choose the point H_1' as one of the origins for the telescopic system, and its conjugate L, with respect to that system, as the other origin.

Let a ray PA (fig. 6.20) be incident on the first system in a direction parallel to the axis. It will leave this system at A′ and pass through F_1', the image focal point of the first system. The ray $A'F_1'$ then passes through the

telescopic system and finally emerges as the ray F′Q′. The points F_1' and F′ are conjugate points for the telescopic system, and if PA be produced to cut F′Q′ in Q′, this point lies on the image principal plane of the combined system (cf. p. 99). We must locate H′ to find this principal plane. Since the ray PA finally leaves the combined system as the ray F′Q′, F′ is the image focal point of the combination, and we must find the value of F′H′ to determine f'. Since F_1' and F′ are conjugate points for the telescopic system, we have $l_2 = -H_1'F_1'$, $l_2' = LF'$, where the suffix 2 applies to this system. Hence

$$LF' = -aH_1'F_1' = -af_1'. \quad . \quad . \quad 6.64$$

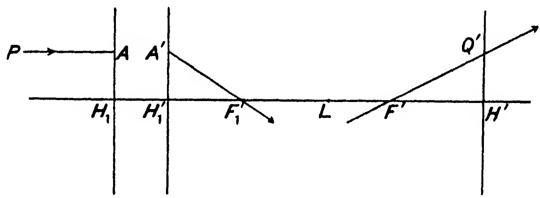

Fig. 6.20

We have also from (6.52), since $\tan\theta' = \tan Q'F'H' = \dfrac{H'Q'}{H'F'}$, and $\tan\theta = \dfrac{A'H_1'}{H_1'F_1'}$,

$$\frac{\tan\theta'}{\tan\theta} = \frac{H_1'F_1'}{H'F'} = \frac{f_1'}{-f'} = \frac{b}{a}. \quad . \quad . \quad 6.65$$

Thus

$$f' = -\frac{a}{b}f_1'. \quad . \quad . \quad . \quad . \quad 6.66$$

We can locate H′ with respect to L by means of

$$\begin{aligned} LH' &= LF' + F'H' \\ &= -af_1' + \frac{a}{b}f_1' \\ &= \left(\frac{1}{b} - 1\right)af_1'. \quad . \quad . \quad . \quad 6.67 \end{aligned}$$

A simple way to find the other principal and focal points is to trace a ray parallel to the axis backward through the system. Suppose that the reversed ray P′Q′ parallel to the axis is incident upon the telescopic system and that RA′ emerges from it to fall on the image principal plane at A′, finally emerging from the combination along AF_1 (fig. 6.21).

As before, we obtain the position of the principal plane by producing P′Q′ to cut F_1A in Q. The plane through

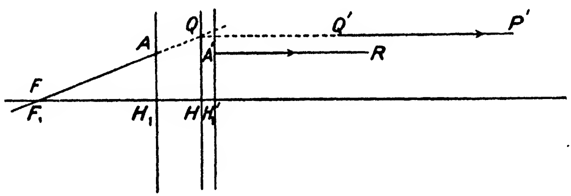

Fig. 6.21

Q perpendicular to the axis is the required plane, and the point of intersection with the axis H is the object principal point.

The ratio of the distances of Q′P′ and A′R from the axis is the linear magnification of the telescopic system $\frac{y'}{y}$.

$$\text{Thus } \frac{QH}{A'H_1'} = b \text{ and hence } \frac{QH}{AH_1} = \frac{HF_1}{H_1F_1} = b.$$

Thus since F_1 is the principal focus of the combination as well as that of the first system,

$$f = bf_1. \quad \dots\dots \quad 6.68$$

The distance of H from H_1 is obtained from

$$HH_1 = HF_1 - H_1F_1 = f - f_1 = f_1(b - 1).$$

If this is positive H is to the right of H_1, and to the left if negative. The focal lengths given by (6.66) and (6.68) are equal since the combined system is surrounded by air. This requires that $\frac{b^2}{a} = -1$, and we have seen that this is the case for a telescopic system with the same medium on both sides as is the case in this example.

CHAPTER VII

I. NON-PARAXIAL RAYS. SPHERICAL ABERRATION.
II. NON-PARAXIAL PENCILS. ASTIGMATISM.
III. CURVATURE AND DISTORTION.

I. The Study of Rays lying outside the Paraxial Zone

In the previous chapters all the formulæ have been derived on the assumption that the rays are inclined at small angles to the axis and that they make small angles with the normals to the refracting surfaces. Under these circumstances it was shown that rays from a point are brought together to pass through a second point by reflection or refraction in an optical system. We also assumed that rays from small line or plane objects could be focussed by the various optical systems to give line or plane images, and in support of this assumption we referred to experience with mirrors, lenses, and such complex apparatus as microscopes and cameras.

It is true that there are generally many rays falling on the system satisfying these conditions, and they are sufficient to produce a visible image. But there must also be rays falling at larger angles of incidence which do not satisfy the conditions assumed, and we shall in this chapter consider the paths of such rays.

We shall not be able to give to this study the same generality as was given to the paraxial case in the last chapter, but we shall examine a number of special cases.

It is one of the difficulties of geometrical optics that

the beginner finds it difficult to realize that practical applications can result from the stringent assumptions we have made. Only experience in verifying the formulæ obtained with mirrors and lenses will show their practical value and their limitations, but a study of rays inclined at larger angles throws light on the nature of the assumptions made and removes the suspicion that is likely to be present and, in fact, ought to be present in the mind of the student at the beginning of his study.

The Plane Mirror

As we have seen, the limitations with regard to small angles do not apply in this case. The formulæ obtained are true whatever the size of the angles of incidence.

The Plane Refracting Surface

Let a point source O lie on the normal to a plane refracting surface PA (fig. 7.1). Consider a ray OP, incident at an angle i, and let i' be the angle of refraction.

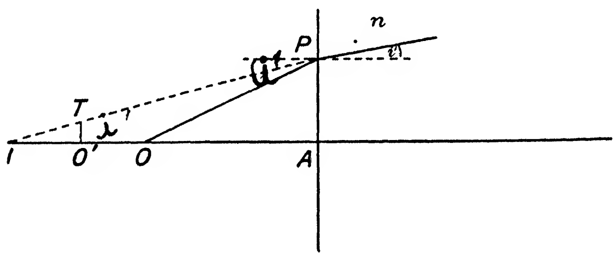

Fig. 7.1

From the figure it may be seen that

$$\text{IA} \tan i' = \text{OA} \tan i, \qquad . \quad . \quad . \quad 7.1$$

and from the law of refraction

$$\sin i = n \sin i',$$

it follows that when O is fixed the position of I depends upon the angle i.

It is only when i is small that I is independent of i.

If we draw a circle with centre A and radius AP on the refracting surface, it is clear that all rays from O to the circumference of this circle will pass through I after refraction. I may be regarded as an image for this particular group of rays.

In most cases considered the angle i may be allowed to vary slightly without causing the rays to deviate much from the direction IP after refraction. Thus the small region near I may be regarded as a kind of image for rays making angles of incidence over the small range i to $i + di$. The difference in this case from that of the paraxial case is that the position of I depends upon i.

If we consider the case of small angles the circle will be drawn with a small radius round A. All angles of incidence for rays from O to points on this circle and within it are small, and the image point O′ will be the same for all. Even when the angle is small, there is a considerable area about A for which this is true, and there will be many rays focussed at O′, which will consequently be bright.

This point O′ is often used as a reference point for the other rays, and we call the distance O′I the longitudinal aberration of the ray PI, while O′T is the transverse aberration.

From $n = \dfrac{\sin i}{\sin i'}$ we deduce $n\mathrm{OP} = \mathrm{IP}$, thus

$$n^2(\mathrm{AP}^2 + \mathrm{AO}^2) = \mathrm{AP}^2 + \mathrm{AI}^2.$$

If we neglect higher powers of AP than the second, we obtain

$$\mathrm{AI} = n\mathrm{AO} + \tfrac{1}{2}\frac{n^2 - 1}{n} \cdot \frac{\mathrm{AP}^2}{\mathrm{AO}}.$$

For small values of AP we have

$$\mathrm{AO}' = n\mathrm{AO},$$

a relation we obtained in Chapter III.

Thus the longitudinal aberration is

$$\mathrm{AI} - \mathrm{AO}' = \tfrac{1}{2} \cdot \frac{n^2 - 1}{n} \cdot \frac{\mathrm{AP}^2}{\mathrm{AO}}. \quad . \quad . \quad 7.2$$

The Convex Spherical Mirror

The reflection of rays at a spherical surface offers another example similar in principle to the case just considered, but modified by the geometry of the surface. We shall consider the case of the convex surface (fig. 7.2).

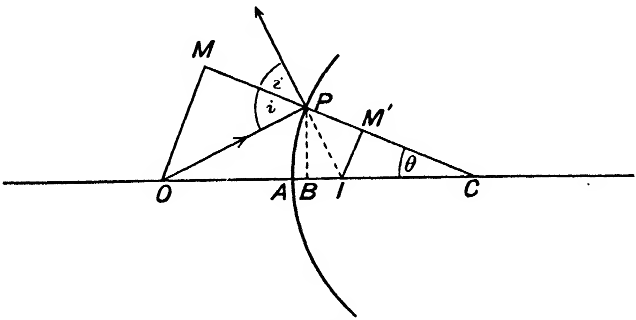

Fig. 7.2

Let a ray OP strike the surface at P at the angle of incidence i and let the normal PC make an angle θ with the axis. Draw OM and IM′ perpendicular to CP.

The triangles COM and CIM′ are similar, and therefore

$$\frac{\mathrm{CM}}{\mathrm{OM}} = \frac{\mathrm{CM}'}{\mathrm{IM}'}.$$

$$\therefore \frac{\mathrm{CP} + \mathrm{OP}\cos i}{\mathrm{OP}\sin i} = \frac{\mathrm{CP} - \mathrm{IP}\cos i}{\mathrm{IP}\sin i},$$

whence

$$\frac{1}{\mathrm{IP}} - \frac{1}{\mathrm{OP}} = \frac{2\cos i}{\mathrm{CP}}. \quad . \quad . \quad . \quad 7.3$$

Let s and s' denote distances measured along the rays of light. In agreement with the notation used in the

paraxial case, we shall measure these distances from the surface, i.e. from P, and we have

$$s = \mathrm{PO}, \quad s' = \mathrm{PI}, \quad \text{and} \quad \mathrm{CP} = r.$$

Thus (7.3) becomes

$$\frac{1}{s'} - \frac{1}{s} = \frac{2\cos i}{r}. \qquad 7.4$$

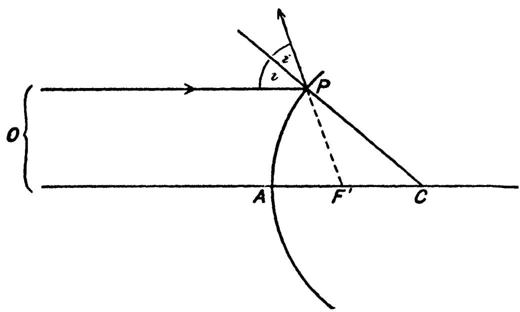

Fig. 7.3

Let us compare this with (2.5), using the values of f and f' given by (2.11), i.e. with

$$\frac{1}{l'} - \frac{1}{l} = \frac{2}{r}. \qquad 7.5$$

(7.4) is the exact equation, (7.5) is the approximation when the angle i is small.

We shall speak of the points O and I as conjugate points; they are, however, merely points of intersection of the particular rays with the axis. We take a point O on the ray, draw a line through it and the centre of the mirror, and note the point of intersection I of this line and the reflected ray. Points on the incident and reflected ray can be associated in pairs in this way, and such pairs we describe as conjugate. If the point O be situated at a great distance from P, the lines OP and OC become parallel (fig. 7.3) and I lies on CO, which is now

the parallel to the incident ray drawn through C. This point is denoted by F′ because it corresponds to the image focal point of the paraxial case. PF′ corresponds to the distance f', and from (7.4) this distance is given by

$$\frac{1}{\mathrm{PF}'} = \frac{2\cos i}{r}. \quad . \quad . \quad . \quad . \quad 7.6$$

Similarly, if the ray after reflection cuts CO at an infinitely distant point, the case becomes that represented in fig. 7.4.

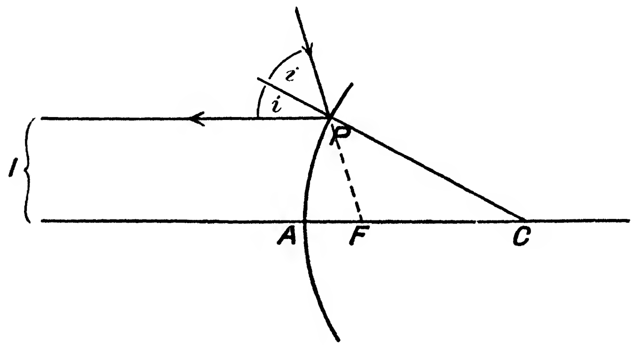

Fig. 7.4

The lines CAI and PI become parallel, and the incident ray when produced cuts CA in F, which is the point O in this case, and is so described because it corresponds to the object focal point. We have, from (7.4),

$$\frac{1}{\mathrm{PF}} = -\frac{2\cos i}{r}. \quad . \quad . \quad . \quad . \quad 7.7$$

From symmetry these points O, I, F and F′ are fixed points so long as the angle i remains the same. Thus in fig. 7.2, if P describes a circle about the point B, the same geometrical relations are preserved, and all rays from O to this circle will, after reflection, pass through I. Thus I is again an image for these rays, and rays with an

incident angle only slightly differing from i will come to a focus near I.

If we describe the distances PF and PF′ as focal distances, we see that a special value of this focal distance belongs to each angle of incidence.

If a ray OP fall at an angle i on the mirror, there will be a particular principal focus corresponding to this direction or, as we may say, appropriate to this particular ray. In order to find this point it is clear from fig. 7.3 that we must draw a line through C parallel to the ray, and the point of intersection with the reflected ray is F′ (fig. 7.5). Similarly, in order to find F we draw through C a parallel to the refracted ray. Its point of intersection with the incident ray is F (fig. 7.4).

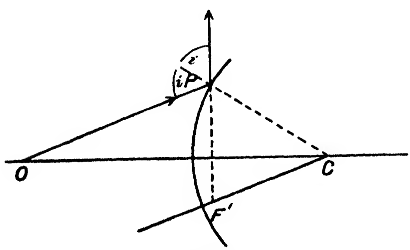

Fig. 7.5

The power in the paraxial case has been seen to be

$$F = -\frac{2}{r},$$

and we may now generalize this expression to suit the present case by writing

$$F = -\frac{2\cos i}{r}, \quad . \quad . \quad . \quad . \quad 7.8$$

which we shall describe as the power for the particular ray in question, or better, for the zone of the optical system on which it falls. Thus the power of the surface can only be regarded as a constant quantity for the paraxial case; in general it varies with the group of rays considered.

The case of the concave mirror will not be separately considered, but the reader will find it a useful exercise

to show that this case is included in the foregoing by preserving the convention with regard to the sign of the radius of curvature which has already been introduced.

The Spherical Aberration for a Spherical Reflecting Surface

In order to find the longitudinal spherical aberration in this case, we require the difference (AI − AO′), where O′ denotes the paraxial image.

From the triangle PCO (fig. 7.2)

$$\frac{\mathrm{CP}}{\mathrm{CO}} = \frac{\sin(i-\theta)}{\sin i}$$

and from PCI

$$\frac{\mathrm{CP}}{\mathrm{CI}} = \frac{\sin(i+\theta)}{\sin i}.$$

If these equations be added, we obtain

$$\frac{1}{\mathrm{CI}} + \frac{1}{\mathrm{CO}} = \frac{2\cos\theta}{r}. \quad . \quad . \quad . \quad 7{\cdot}9$$

If θ is small, I lies at O′, the conjugate to O for paraxial rays. Thus

$$\frac{1}{\mathrm{CO}'} + \frac{1}{\mathrm{CO}} = \frac{2}{r},$$

whence by subtraction

$$\frac{1}{\mathrm{CO}'} - \frac{1}{\mathrm{CI}} = \frac{2}{r}(1-\cos\theta). \quad . \quad . \quad 7.10$$

We shall suppose that the angle θ is small enough to make it possible to neglect powers of θ higher than θ^2.

In the paraxial case we neglected powers higher than θ, so that we are now proceeding to the next order of accuracy.

When $\cos\theta$ is expanded in terms of θ, we find

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots .$$

In the previous case we wrote $\cos\theta = 1$; we propose now to write

$$\cos\theta = 1 - \tfrac{1}{2}\theta^2.$$

Thus from (7.10)

$$\frac{1}{CO'} - \frac{1}{CI} = \frac{\theta^2}{r},$$

$$CI - CO' = \frac{\theta^2}{r} \, . \, CO' \, . \, CI.$$

This equation shows that the difference between CI and CO′ is of the second degree in θ, so that if we write $CO' = CI$ on the right-hand side of this equation, the result will still be accurate to this order. Thus

$$CI - CO' = \frac{\theta^2}{r} \, . \, CO'^2. \qquad . \quad . \quad 7.11$$

This is the expression for the longitudinal spherical aberration in cases where higher powers of θ than the square may be neglected.

A similar treatment may be applied to the concave surface, and again we must pay attention to the sign of r. (7.11) is the formula for both convex and concave mirrors.

Refraction at a Spherical Surface

Let the spherical surface AP separate media of refractive indices n and n'. If reference be made to fig. 7.6, it will be seen that

$$\frac{CM}{CM'} = \frac{OM}{IM'},$$

the notation being similar to that of the previous case. Since $n \sin i = n' \sin i'$, it is easy to deduce

$$\frac{n'}{IP} + \frac{n}{OP} = \frac{n' \cos i' - n \cos i}{r}. \qquad . \quad 7.12$$

The paraxial equation (3.19) is

$$\frac{n'}{\mathrm{O'P}} + \frac{n}{\mathrm{OP}} = \frac{n' - n}{r}. \quad . \quad . \quad . \quad 7.13$$

As in the previous example, the power must take into consideration the inclination of the rays to the normal and becomes in this case

$$\mathrm{F} = \frac{n' \cos i' - n \cos i}{r}. \quad . \quad . \quad 7.14$$

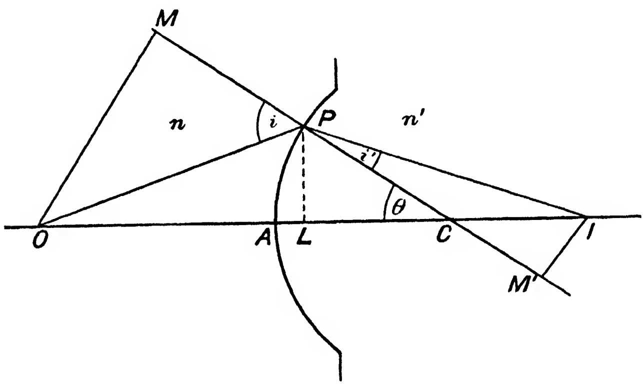

Fig. 7.6

The Principal Foci

These points are obtained, as in the case of the mirrors, by drawing parallels through C to the incident and refracted rays. The points of intersection of these parallels with the rays give F and F′, as shown in fig. 7.7. An interesting extension of the paraxial equations referred to the foci can be obtained. In that case we had $xx' = ff'$. Writing (7.12) in the form

$$\frac{n'}{\mathrm{IP}} + \frac{n}{\mathrm{OP}} = \mathrm{F},$$

it is evident that when OP is very great, IP becomes F′P, and consequently

$$F'P = \frac{n'}{F}. \qquad \ldots \ldots \qquad 7.15$$

In the same way,

$$FP = \frac{n}{F}. \qquad \ldots \ldots \qquad 7.16$$

A ray from O falling on the surface at P (fig. 7.7) intersects OC at I after refraction. Thus O and I are conjugate points related by equation (7.12).

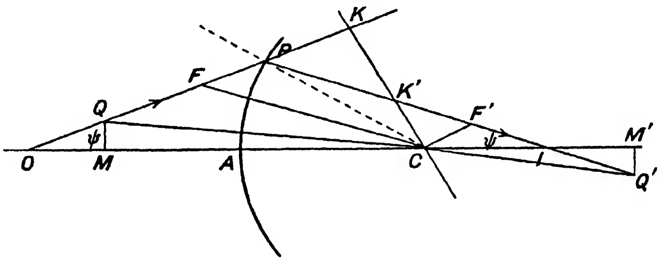

Fig. 7.7

The triangles OFC and IF′C are similar by construction, CF being parallel to IP and CF′ parallel to OP. Thus

$$\frac{OF}{FC} = \frac{F'C}{IF'}$$

or

$$OF \,.\, IF' = FC \,.\, F'C. \qquad \ldots \ldots \qquad 7.17$$

OF and IF′ are the distances of object and image from the principal foci respectively, and FC = PF′, the image focal distance for the inclination *i*, and F′C = PF, the object focal distance. We can write

$$OF \,.\, IF' = \frac{nn'}{F^2}, \qquad \ldots \ldots \qquad 7.18$$

and this is analogous to the paraxial relation

$$\mathrm{OF} \,.\, \mathrm{O'F'} = ff'.$$

The Nodal Points in the General Case

If any line is drawn through C, e.g. OCI in fig. 7.7, it will cut the incident and refracted ray in two conjugate points, O and I.

Draw the bisector of the angle FCF′ and let it cut the rays in K and K′. The points are conjugate points, and since $\angle \mathrm{FCK} = \mathrm{F'CK'} = \mathrm{FKC}$, it follows that $\mathrm{KF} = \mathrm{CF} = \mathrm{PF'} = \dfrac{n'}{\mathrm{F}}$ (7.15), and similarly $\mathrm{K'F'} = \dfrac{n}{\mathrm{F}}$.

In (7.18) we have a relation satisfied by any pair of conjugate points O and I on the same ray. Thus

$$\mathrm{OF} \,.\, \mathrm{IF'} = \mathrm{KF} \,.\, \mathrm{K'F'} = \frac{nn'}{\mathrm{F}^2}.$$

$$\therefore\ (\mathrm{OK} - \mathrm{KF})(\mathrm{IK'} - \mathrm{K'F'}) = \mathrm{KF} \,.\, \mathrm{K'F'},$$

whence

$$\frac{\mathrm{K'F'}}{\mathrm{IK'}} + \frac{\mathrm{KF}}{\mathrm{OK}} = 1.$$

This may be written

$$\frac{n/\mathrm{F}}{\xi'} + \frac{n'/\mathrm{F}}{\xi} = 1, \qquad \ldots \qquad 7.19$$

where ξ and ξ' are measured along the rays from the points K and K′.

If this be compared with (6.28), it will be seen that K and K′ correspond to the nodal points.

Spherical Aberration for Refraction at a Spherical Surface

Referring to fig. 7.6, we have

$$\frac{\sin i}{\sin \theta} = \frac{\mathrm{OC}}{\mathrm{OP}}, \quad \frac{\sin i'}{\sin \theta} = \frac{\mathrm{IC}}{\mathrm{IP}}.$$

Hence

$$n\frac{\text{OC}}{\text{OP}} = n'\frac{\text{IC}}{\text{IP}}. \qquad . \quad . \quad . \quad . \quad . \qquad 7.20$$

We write $\mu = \frac{n'}{n}$, and in the case of an incident ray in air, $\mu = n'$, where n' is the refractive index of the second medium

$$\text{OC} = \text{OA} + \text{AC}, \quad \text{IC} = \text{IA} - \text{AC}.$$

We shall simplify the writing of the equations if we place

$$\text{AO} = l, \quad \text{AI} = v.$$

Then

$$\text{OC} = l + r, \quad \text{IC} = v - r.$$

Since

$$\text{OP}^2 = \text{OC}^2 + \text{PC}^2 - 2\text{OC} \,.\, \text{PC} \cos\theta$$

and

$$\text{IP}^2 = \text{IC}^2 + \text{PC}^2 + 2\text{IC} \,.\, \text{PC} \cos\theta,$$

we obtain from (7.20) by squaring:

$$\frac{(l+r)^2}{(l+r)^2 + r^2 - 2r(l+r)\cos\theta} = \mu^2 \frac{(v-r)^2}{(v-r)^2 + r^2 + 2r(v-r)\cos\theta}. \qquad 7.21$$

This equation is exact, but if we now substitute $\cos\theta = 1 - \frac{1}{2}\theta^2$, making the approximation used in the case of the reflecting surface, the denominator of the left-hand side becomes $l^2\left\{1 + r\frac{(l+r)}{l^2}\theta^2\right\}$, and of the right-hand side $v^2\left\{1 - \frac{r(v-r)}{v^2}\theta^2\right\}$.

Thus, taking the square root of each side, we obtain:

$$\mu\frac{(v-r)}{v}\left\{1 + \frac{r(v-r)}{2v^2}\theta^2\right\} = \frac{(l+r)}{l}\left\{1 - \frac{r(l+r)}{2l^2}\theta^2\right\},$$

whence

$$\mu\left(\frac{1}{r}-\frac{1}{v}\right)+\mu\frac{(v-r)^2}{2v^3}\theta^2=\left(\frac{1}{r}+\frac{1}{l}\right)-\frac{(l+r)^2}{2l^3}\theta^2. \quad 7.22$$

Neglecting θ^2, we obtain again the paraxial case (3.19)

$$\mu\left(\frac{1}{r}-\frac{1}{l'}\right)=\frac{1}{r}+\frac{1}{l},\quad \mu=\frac{n'}{n}, \quad . \quad 7.23$$

where v is now l', the image distance.

We have taken the positive sign in the square roots, and this is now justified in that it gives the correct sign for the paraxial equation.

From (7.22) and (7.23)

$$\mu\left(\frac{1}{v}-\frac{1}{l'}\right)=\tfrac{1}{2}\theta^2\left\{\mu\frac{(v-r)^2}{v^3}+\frac{(l+r)^2}{l^3}\right\}$$

$$=\tfrac{1}{2}\theta^2\cdot\left(\frac{l+r}{l}\right)^2\left\{\frac{1}{l}+\frac{\mu}{v}\left(\frac{1/r-1/v}{1/r+1/l}\right)^2\right\}. \quad 7.24$$

To the order of approximation we require we can make use of (7.23) to modify the second term in the bracket, which becomes $\frac{1}{\mu l'}$, where v is replaced by l', from which it differs by a quantity of the second order in θ (7.24), so that to this order of magnitude (7.24) is replaced by

$$\mu\left(\frac{1}{v}-\frac{1}{l'}\right)=\tfrac{1}{2}\theta^2\left(\frac{l+r}{l}\right)^2\left(\frac{1}{l}+\frac{1}{\mu l'}\right). \quad 7.25$$

The left-hand side gives the change in value of $\frac{1}{l'}$, due to making a closer approximation than that given by the paraxial results. We can thus write $\left(\frac{1}{v}-\frac{1}{l'}\right)=\Delta\left(\frac{1}{l'}\right)$, and, as it is more convenient, we shall introduce reciprocals of the various quantities

$$\frac{1}{l}=\mathrm{L},\quad \frac{1}{l'}=\mathrm{L}',\quad \frac{1}{r}=\mathrm{R}. \quad . \quad . \quad 7.26$$

Thus the above equation can be rewritten in the form

$$\mu\Delta L' = \tfrac{1}{2}\frac{\theta^2}{\mu R^2}(R+L)^2(\mu L + L'). \quad . \quad 7.27$$

The equation (7.23) may be written in the form

$$\mu(R - L') = R + L,$$

and thus (7.27) can be rewritten in the form

$$\Delta L' = \frac{\mu - 1}{2\mu^3}\cdot\frac{\theta^2}{R^2}\cdot(R+L)^2(R+\overline{\mu+1}L). \quad 7.28$$

The longitudinal aberration is $(v - l')$, and this may be obtained at once from $\Delta L'$, for

$$\Delta L' = \frac{1}{v} - \frac{1}{l'} = -\frac{v - l'}{vl'} = -\frac{\Delta l'}{l'^2}. \quad 7.29$$

to the degree of approximation we are considering; for by (7.25) it is clear that v differs from l' by a quantity proportional to the square of θ, and thus we can write $vl' = l'^2$ in the denominator of (7.29).

In the case of an optical instrument we require most frequently the aberration for the extreme ray, and if OP denotes this ray, the perpendicular PL is the radius of the aperture of the refracting surface. We are concerned here only with symmetrical pencils of rays, and we are neglecting higher powers of θ than the square so that we can write $PL^2 = r^2\theta^2$ and, denoting PL by a, we can express (7.28) in the form

$$\begin{aligned}\Delta L' &= \frac{\mu - 1}{2\mu^3}(R+L)^2(R+\overline{\mu+1}L)a^2 \\ &= ka^2 \text{ (say)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 7.30\end{aligned}$$

Aplanatic Points

By referring to the exact equation (7.21), we can derive an interesting result by noting that when

$$(l + r) = -\mu^2(v - r) \quad . \quad . \quad . \quad 7.31$$

θ does not enter into the relation, and for a given value of l, the value of v is independent of θ. Thus all the rays in this case pass through a point I after refraction. The surface focusses all the rays from O at a single point I. These points are called aplanatic points for the surface.

We can find the values of l and v for which this holds from (7.31) and (7.21). The latter equation gives

$$1+\left(\frac{r}{l+r}\right)^2=\frac{1}{\mu^2}+\frac{1}{\mu^2}\left(\frac{r}{v-r}\right)^2,$$

and in combination with (7.31) we find

$$1+\left(\frac{r}{l+r}\right)^2=\frac{1}{\mu^2}+\mu^2\left(\frac{r}{l+r}\right)^2$$

or

$$l=-r\pm\mu r,\quad v=r\mp\frac{r}{\mu}. \qquad 7.32$$

Consider first $l=-r+\mu r$, $v=r-\dfrac{r}{\mu}$.

In this case the point O is to the left of C at a distance μr from it, and I is to the left of C at a distance $\dfrac{r}{\mu}$ from it. If we consider the alternative case

$$l=-r-\mu r,\quad v=r+\frac{r}{\mu},$$

we see that it corresponds to the case when O lies to the right of C at a distance μr, and I lies to the right at a distance $\dfrac{r}{\mu}$. The two cases are thus symmetrical about the centre of the spherical surface. It is clear from symmetry that if one point satisfies the conditions the other does so as well. Thus the pair of points situated on the diameter of a sphere at distances μr and $\dfrac{r}{\mu}$ from the centre and on the same side of it are aplanatic.

Young has given a simple construction which demonstrates the existence of these points.

Let the refracting surface be of radius r, with its centre at C (fig. 7.8), Describe spheres about the centre C of radii μr and $\frac{r}{\mu}$. The figure is drawn for the case $\mu > 1$.

Let AP be a ray cutting the sphere in P, and let it be produced to cut the larger sphere in B. Let BC cut the

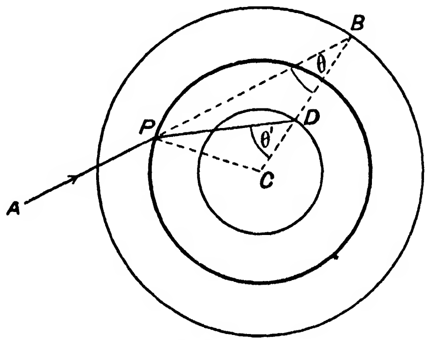

Fig. 7.8

smaller sphere in D. Then PD denotes the refracted ray. This may be shown by proving that

$$\sin \mathrm{CPB} = \mu \sin \mathrm{CPD}.$$

The triangles PCD and PCB have the angle C common to both, and by construction

$$\frac{\mathrm{CP}}{\mathrm{CD}} = \frac{\mathrm{CB}}{\mathrm{CP}} = \mu,$$

so that the triangles are similar.

Thus

$$\angle \mathrm{CDP} = \angle \mathrm{CPB},$$

and hence, since

$$\frac{\sin \mathrm{CDP}}{\sin \mathrm{CPD}} = \frac{\mathrm{CP}}{\mathrm{CD}} = \mu,$$

it follows that

$$\sin \mathrm{CPB} = \mu \sin \mathrm{CPD}.$$

This result is true for all positions of P, and it therefore follows that any ray directed towards B is refracted through D.

The converse is true that rays directed from D appear to come from B. These two points are aplanatic points, and their distances from the centre are μr and $\frac{r}{\mu}$, as we saw above.

An interesting relation holds between the inclinations of the rays to the line through the points, i.e. between the angles CBP and CDP.

We have

$$\frac{\sin \mathrm{CDP}}{\sin \mathrm{CBP}} = \frac{\sin \mathrm{CPB}}{\sin \mathrm{CPD}} = \mu = \frac{n'}{n}.$$

If these inclinations be denoted by θ and θ',

$$\frac{\sin \theta'}{\sin \theta} = \frac{n'}{n}. \qquad \dots \dots \quad 7{\cdot}33$$

This is true for all inclinations θ and θ', of rays which pass through the aplanatic points.

It is instructive to contrast this relation with (6.25). In that case we were considering a relation where it was assumed that an image could be formed, and the result obtained is subject to the limitations implied in that assumption. The sine relation (7.33) has been obtained for any angles θ and θ', but for rays through the aplanatic points only.

In the present case any radius cuts the two auxiliary spheres in a pair of aplanatic points. Thus if we take an area of the surface about B, the rays directed to it will, after refraction, be brought to a focus at the corresponding area about D. The two areas stand in the relation of object and image for the refracting surface. They are

not plane as in the previous cases considered, but they are exactly similar.

There is a very interesting relation between these two areas, which is an example of a result we shall consider later in a paragraph on the law of sines.

If the areas are S and S′,

$$\frac{S'}{S}=\frac{CD^2}{CB^2}=\frac{1}{\mu^4}.$$

Thus from (7.33),

$$n^2\sin^2\theta S = n'^2\sin^2\theta' S'. \quad . \quad . \quad 7.34$$

Spherical Aberration in the case of a Lens

The lens is represented in fig. 7.9 as having bounding surfaces with positive curvatures, but the results obtained will be general if the appropriate signs of the quantities are applied. Let the refractive indices of the media be n, n' and n''.

We shall denote the ratios $\frac{n'}{n}$ by μ_1 and $\frac{n''}{n'}$ by μ_2

$$\mu_1=\frac{n'}{n},\ \mu_2=\frac{n''}{n'}. \quad . \quad . \quad . \quad 7.35$$

In the important case of a lens situated in air, $\mu_1=\frac{1}{\mu_2}=n'$.

Let O denote a source of rays situated on the axis of the lens, and let one of the rays be represented by OP. We shall treat this ray as the extreme ray entering the system, and we shall describe the aberration in this case as the aberration of the lens.

We shall neglect powers of ∠POM beyond the second. If O′ is the position of the paraxial image of the lens, we require the magnitude of O′I, which is described as the longitudinal aberration.

The ray OP after refraction at the first surface passes along PP′ and is directed towards I_1 and at P′ is refracted again towards I.

Let O_1 denote the paraxial image for the first refraction, so that it is the point conjugate to O' for refraction at the second surface. The displacement of I from O' can be regarded as resulting from two causes: in the first place it is due to the fact that the point I_1 is displaced from the point O_1 and in the second it is due to spherical aberration at the second surface.

We have thus to find the displacement of O' due to a displacement from O_1 to I_1. Let l_1' denote the distance AO_1. Then with the same notation as before, using capital

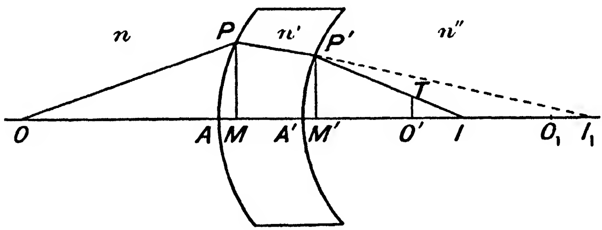

Fig. 7.9

letters for the reciprocals of quantities represented by small letters, we find from (7.30)

$$\Delta L_1' = \frac{\mu_1 - 1}{2\mu_1^3}(R_1 + L_1)^2(R_1 + \overline{\mu_1 + 1}L_1)a_1^2$$
$$= k_1 a_1^2 \text{ (say)}, \quad . \quad . \quad . \quad . \quad 7.36$$

where

$$PM = a_1.$$

From (7.29) we see that the displacement O_1I_1 is of magnitude $-l_1'^2 \Delta L_1'$.

The equation relating the positions of conjugate points for refraction at the second surface is

$$\frac{n''}{l_2'} + \frac{n'}{l_2} = \frac{n'' - n'}{r_2},$$

where the distances are measured from A'

If l_2 is changed by the amount Δl_2, the change in l_2' is given by

$$\Delta l_2' = -\frac{l_2'^2}{l_2^2} \cdot \frac{\Delta l_2}{\mu_2}.$$

In this formula l_2 is positive when measured towards the left, and consequently a positive value of Δl_2 means a displacement towards the left.

In the calculation of the displacement from (7.36) it must be remembered that the change required is a change in the position of the image point and a displacement towards the right is positive. Thus in the displacement $-l_1'^2 \Delta L_1'$ we must write

$$\Delta l_2 = l_1'^2 \Delta L_1'.$$

Thus the corresponding value of $\Delta l_2'$ is

$$\begin{aligned} \Delta l_2' &= -\frac{l_2'^2 l_1'^2}{\mu_2 l_2^2} \Delta L_1' \\ &= -\frac{k_1}{\mu_2} l_2'^2 \frac{l_1'^2}{l_2^2} a_1^2. \end{aligned}$$

l_2' is the distance of the final image from A′, and we shall follow the usual custom of denoting it by l'.

Thus

$$\Delta l_2' = -\frac{k_1}{\mu_2} l'^2 \frac{l_1'^2}{l_2^2} a_1^2. \quad . \quad . \quad . \quad 7{\cdot}37$$

The second part of the displacement is due to the spherical aberration at the second surface and is given by $-l_2'^2 \Delta L_2'$, where

$$\begin{aligned} \Delta L_2' &= \frac{\mu_2 - 1}{2\mu_2^3} (R_2 + L_2)^2 (R_2 + \overline{\mu_2 + 1} L_2) a_2^2 \\ &= k_2 a_2^2 \text{ (say)}. \quad . \quad . \quad . \quad . \quad 7{\cdot}38 \end{aligned}$$

L_2 denotes the reciprocal of the distance l_2 of O_1 from A′.

Strictly, the object in this case is at I_1 and not at O_1, but the difference between $A'I_1$ and $A'O_1$ is proportional

to the square of the inclination of the ray, so that the change made in (7.38), which is already proportional to the square of the inclination, would be of a higher order than that we are considering.

We have a further relation between the quantities concerned here, for if d denote the thickness AA′ of the lens,

$$AO_1 - A'O_1 = d,$$

$$l_1' + l_2 = \frac{1}{L_1'} + \frac{1}{L_2} = d. \qquad 7.39$$

These equations are sufficient to determine the aberration for the lens, the magnitude being

$$\Delta l_2' - l'^2 \Delta L_2 = -l'^2\left(\frac{k_1}{\mu_2}\frac{l_1'^2}{l_2^2}a_1^2 + k_2 a_2^2\right). \qquad 7.40$$

The Spherical Aberration in the case of a Thin Lens

This case presents some simplifying features especially in the case of a thin lens situated in air, which we shall consider. In this case $n = n'' = 1$.

In fig. 7.9 A and A′ now become coincident.

Thus $a_1 = a_2 = a$ (say) and $l_2 = -l_1'$, these symbols denoting the same distance $A'O_1$ or AO_1 but with opposite sign conventions. Thus $L_2 = -L_1'$.

Consequently

$$k_1 = \frac{n'-1}{2n'^3}(R_1 + L)^2(R_1 + \overline{n'+1}L)$$

and

$$k_2 = -\tfrac{1}{2}n'^2(n'-1)(R_2 + L_2)^2\left(R_2 + \frac{n'+1}{n'}L_2\right).$$

L_2 is a quantity serving in an intermediate capacity like the point O_1, and it will be eliminated. This is made possible by the relation

$$\frac{n''}{l'} + \frac{n'}{l_2} = \frac{n''-n'}{r_2},$$

which may now be written

$$L' + n'L_2 = (1 - n')R_2.$$

Thus the value of k_2 becomes

$$k_2 = -\frac{n' - 1}{2n'^2}(R_2 - L')^2(R_2 - \overline{n' + 1}L').$$

Thus from (7.40) the total displacement of O′ is $\Delta l'$, where

$$\begin{aligned} \Delta l' &= -l'^2(n'k_1 + k_2)a^2 \\ &= -l'^2a^2\frac{n' - 1}{2n'^2}\{(R_1 + L)^2(R_1 + \overline{n' + 1}L) \\ &\qquad - (R_2 - L')^2(R_2 - \overline{n' + 1}L')\}. \quad . \quad . \quad 7.41 \end{aligned}$$

The result is expressed in terms of the paraxial distances of the object and image, and gives the distance of the intersection of the extreme ray with the axis from the paraxial image. We can further eliminate L or L′ by means of the lens formula, which, expressed in terms of reciprocals of the distances, is

$$L' + L = (n' - 1)(R_1 - R_2). \quad . \quad 7.42$$

The Aberration in the case of a Thin Lens for Parallel Rays incident on the Lens Margin

This case is of importance on account of its practical application. It is easily studied by means of (7.41), for we have now $L = 0$.

From (7.42) $L' = (n' - 1)(R_1 - R_2)$, and this is the reciprocal of the focal length, which will be denoted by F. Thus from (7.41)

$$\frac{\Delta l'}{l'^2} = a^2\frac{n' - 1}{2n'^2}\{(R_2 - F)^2(R_2 - \overline{n' + 1}F) - R_1^3\}. \quad 7.43$$

We shall express this result in terms of the ratio of the radii of curvature and of F.

Writing $\sigma = \dfrac{r_1}{r_2} = \dfrac{R_2}{R_1}$, we have

$$R_1 = \frac{F}{(n' - 1)(1 - \sigma)}, \quad R_2 = \frac{\sigma F}{(n' - 1)(1 - \sigma)}.$$

After some algebraical manipulations, which will be shortened by substituting $\zeta = (1 - \sigma)$, it is found possible to rewrite (7.43) in the form

$$\Delta l' = -\frac{a^2 F}{2n'(n' - 1)^2(1 - \sigma)^2} \times \{2 - 2n'^2 + n'^3 + (n' + 2n'^2 - 2n'^3)\sigma + n'^3\sigma^2.\} \qquad 7.44$$

a denotes the radius of the lens, and the sign of σ is determined by its character, it being negative for a double convex or double concave lens.

The Sign of the Aberration for Parallel Rays

In practice the value of n' lies between $1 \cdot 5$ and 2. In this case the roots of the quantity within the large bracket of (7.44) are imaginary. When σ is zero its value $(2 - 2n'^2 + n'^3)$ is positive, lying in the region $\frac{7}{8}$ to 2, so that the value of the bracket is always positive. For if it became negative for any value of σ, it would have to vanish in passing from its positive value for $\sigma = 0$ to the negative value, i.e. it would have a real root. Thus the sign of $\Delta l'$ depends upon that of F. For a double convex lens this is positive, and thus $\Delta l'$ is negative. This means that parallel rays falling near the margin of the lens cut the axis at a point nearer the lens than the focal point. A similar remark holds for the double concave lens.

*

The Minimum Aberration

For a given aperture and focal length we may inquire what ratio of the radii of curvature will give the least aberration.

We have to determine the minimum value of (7.44) by the usual method.

The calculation gives only one value of σ for which the differential coefficient vanishes, and this is

$$\sigma = \frac{2n'^2 - n' - 4}{n'(2n' + 1)}. \qquad 7.45$$

The value of $\Delta l'$ is infinite for $\sigma = 1$, and since there is only one turning value, $\Delta l'$ must be a minimum at that value. Thus (7.45) gives the value of σ for which the spherical aberration is a minimum.

If $n' = 1{\cdot}5$, the minimum value occurs for $\sigma = -\frac{1}{6}$. Thus r_1 and r_2 are of opposite signs and $r_2 = 6r_1$, numerically, i.e. the light falls first on the more curved surface of the lens. Such a lens is called a crossed lens, and the aberration is $-\frac{15}{14}a^2\mathrm{F}$ for a lens of material with refractive index $1{\cdot}5$.

The spherical aberration for a plano-convex lens with its curved surface turned toward the light is $-\frac{7}{6}a^2\mathrm{F}$. This value may be obtained from (7.44) by the substitutions $n' = 1{\cdot}5$, $\sigma = \infty$. Thus the difference in the two cases is not very great. It is, however, much easier to construct a plano-convex lens than a crossed lens, and for this reason and the fact that the aberration is sufficiently near the minimum the simpler lens is much used in optical instruments.

Some general remarks on the diminution of spherical aberration are of assistance in association with these calculations. In the case of a prism we are familiar with the conditions under which minimum deviation is produced. In that case each surface takes the same share in the production of the deviation, and we make use of this position of the prism in obtaining a pure spectrum.

The important point is that if a pencil of rays is focussed to a point before the intervention of the prism, it is almost exactly focussed to a point after the prism is interposed. The reason is that the amount of deviation varies very little near the minimum, and for an appreciable range of the angle of inclination on either side of the central ray the inclinations of the rays to one another remain almost the same after refraction as before.

In the crossed lens which we have shown to produce minimum spherical aberration in a bundle of parallel rays the inclination of a ray to the surface is the same on incidence as on emergence. If a drawing be made to scale, it will be seen that this condition can be obtained when the parallel rays strike the more curved surface. In cases where the crossed lens is not used the result will be the better the nearer the condition of equal division of refraction at the surfaces is satisfied. This may be illustrated by a drawing to illustrate the case of the plano-convex lens we have described. The curved surface must be placed towards the incident parallel rays in this case, and it becomes clear that with the lens the other way round the angles of incidence and emergence are far from equal, all the deviation being produced by the curved surface.

If the object lies near the principal focus the image exhibits less spherical aberration with the flat surface of the lens towards the object. In this case the emergent rays are almost parallel to the axis and the work of refraction is again shared between the two surfaces.

Spherical aberration can be further reduced by replacing a single lens by an equivalent system of several.

It can also be reduced in a single lens of given focal length by increasing the refractive index.

Thus if we take the case when the focal length is 1 metre and the value of a (equation 7.44) is 10 cm. with $n' = 1{\cdot}5$, the minimum value with $\sigma = -\frac{1}{6}$ gives for $\Delta l'$ the value $1{\cdot}07$ cm., and with $n' = 2$, $\sigma = +\frac{1}{5}$, $\Delta l'$ is diminished to $0{\cdot}44$ cm.

Transverse Aberration

We have obtained formulæ in several cases for the longitudinal aberration of a ray but have not considered transverse aberration, which we mentioned at the beginning of the chapter. We shall now consider an example of this in order to obtain a result which we shall need as an illustration when we come to study optical instruments.

In fig. 7.9 the final emergent ray is P′I and O′ is the image point for paraxial rays. Draw O′T perpendicular to the axis, cutting P′I in T so that O′T is the transverse aberration according to the definition.

The formula (7.41) gives us the value of O′I. We can write this expression in the form

$$\Delta\left(\frac{1}{l'}\right) = -\mathrm{K}a_1^2,$$

where the value of K is obvious from the formula and a_1 is the radius of the aperture of the first surface of the lens.

$$\Delta l' = \mathrm{O'I} = \mathrm{K}a_1^2 l'^2$$

and

$$\frac{\mathrm{O'T}}{a_2} = \frac{\mathrm{O'I}}{\mathrm{M'I}} = \frac{\Delta l'}{l'}$$

very nearly, where in writing $\mathrm{M'I} = l'$ we neglect small quantities not affecting the result to the order of magnitude we are considering. Thus

$$\mathrm{O'T} = \mathrm{K}a_1^2 a_2 l', \qquad . \quad . \quad . \quad 7.46$$

a_2 denoting P′M′, the radius of the aperture of the final surface.

Now the ratio of a_2 to a_1 is some finite quantity, e.g. in a thin lens where P and P′ are approximately coincident, this ratio is unity. Thus (7.46) gives the result

$$\mathrm{O'T} \propto a_1^3, \qquad . \quad . \quad . \quad . \quad 7.47$$

or the transverse aberration is proportional to the cube of the aperture.

Aberration Curves

The examples considered in this chapter have shown that the aberration depends on the point at which the incident ray strikes the system. A graphical represen-

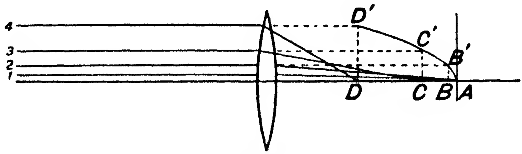

Fig. 7.10

tation is very helpful as an aid to visualizing the nature and extent of the aberration in particular cases.

In fig. 7.10 the aberrations for a system of parallel rays are shown. Ray 1 is represented as occupying the

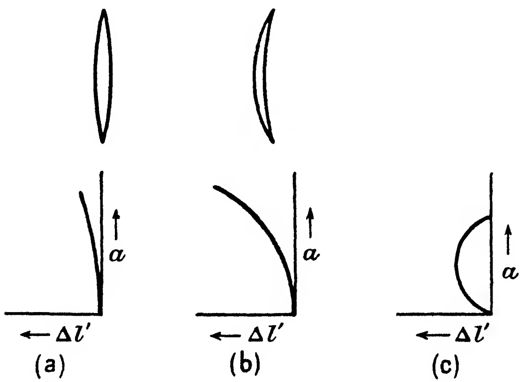

Fig. 7.11

paraxial region, and after passing through the lens it cuts the axis at the principal focus denoted by A. Rays falling at greater distances cut the axis at different points B, C, D, and the distances AB, AC and AD measure the aberrations corresponding to refraction through the system

at the various distances from the axis. We can regard the system as divided into small zones, each with its particular aberration. If we plot the distances of these zones from the axis, i.e. BB′, CC′, DD′, against the aberrations AB, AC, AD, we obtain a curve which may be regarded as describing the aberration of the systems. In the figure this is represented by the curve AB′C′D′. We give in fig. 7.11 the curves for three cases as an illustration: (*a*) is intended to illustrate the crossed lens with minimum aberration, and (*b*) the case of a meniscus lens. It is possible that the aberration for a particular zone may vanish when we obtain the case illustrated in (*c*), where the curve bends back and cuts the axis at the corresponding point.

II. Small Pencils of Rays lying outside the Paraxial Region

We have seen that a small pencil of rays starting from a point on the axis of an optical system is brought to a point focus after reflection or refraction by the system provided that the rays are inclined at a small angle to the axis. When the pencil falls upon the system so that the angles made by its rays are not small, the rays do not, in general, all pass through the same point after emerging from the system. There is no point common to all the rays in the image space. The pencil in this case is described as astigmatic.

A good example of astigmatism is provided by a cylindrical lens. This is a lens bounded by two parallel cylindrical surfaces. We shall consider the case where one of the surfaces is plane and where the lens is thin (fig. 7.12).

In this figure the curved surface is on the left and the plane surface on the right. Let AL_1 denote the axis of the lens and let BAC lie in a plane perpendicular to the axis of the cylinder. Refraction in this plane will be according to the rules of refraction at a surface in the

form of a circle. In the plane at right angles to this, containing the axis of the cylinder, refraction is the same as in a parallel-sided plate. The rays are not deviated in this plane but emerge parallel to the incident direction. Thus, neglecting the effect of spherical aberration, an incident circular cone of rays with the conical axis along AL_1 will emerge from the lens as a cone but with a section no longer circular. In the plane at right angles to the plane of the figure the rays will be focussed at L_1, while in the plane of the figure the rays will go on un-

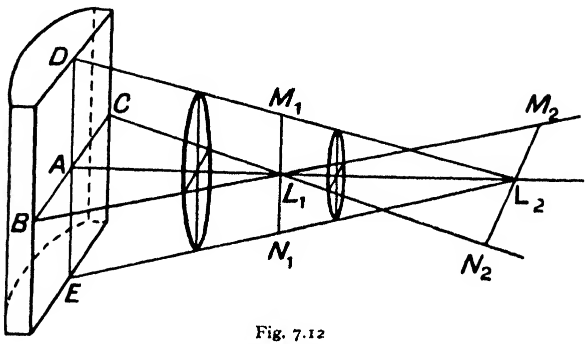

Fig. 7.12

deviated to L_2. Above and below L_1 the rays are focussed at points in the planes parallel to BL_1C, so that all pass through the line M_1N_1 parallel to the axis of the cylinder. The rays likewise all pass through a line through L_2 perpendicular to the axis of the lens.

The bundle in the image space is known as the conoid of Sturm. It has the property of possessing different convergences in two planes mutually perpendicular and that it passes through two mutually perpendicular lines M_1N_1 and M_2N_2.

These lines are described as focal lines, and the distance L_1L_2 is called the astigmatic difference of the lens. This property of the refracted bundle of rays of possessing different curvatures in two planes at right angles is described by the term astigmatism.

Astigmatism of a Reflecting Spherical Surface

Another example of astigmatism is provided by a small pencil of rays incident upon a spherical surface obliquely, i.e. so that the axis of the pencil does not make a small angle with the axis of the surface.

Let O denote a point source of rays which fall obliquely on the surface element MM′ (fig. 7.13). Let OM and OM′ denote the extreme rays and let the angle AOM be θ, while MOM′ is $\Delta\theta$.

Let the extreme rays appear to cross at P_1 after reflection and let them cut the axis in P_2 and P_2'.

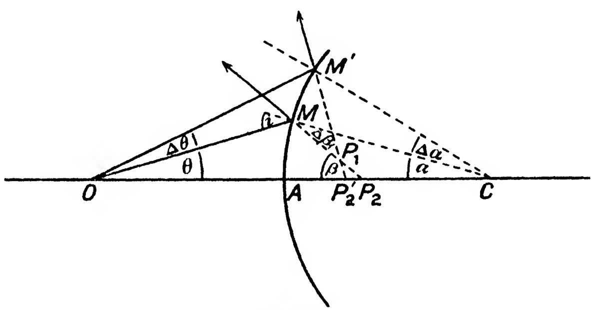

Fig. 7.13

If we imagine the figure to rotate slightly about the axis AO, MM′ will describe a small surface element, and we can suppose that a thin pencil of rays from O falls upon it.

It is clear that all the reflected rays produced will pass through an element of the axis P_2P_2'. The rays will also cross on the locus of P_1 as the rotation is made. If the rotation is small, as it must be if the pencil is thin, the locus of P_1 is approximately a straight line perpendicular to the plane of the figure.

This line and P_2P_2' are the focal lines of the astigmatic reflected pencil, and we shall describe them as the sagittal and tangential focal lines respectively.

Let MP_2 be denoted by s_2, MO by s, and let the radius of the surface be r.

From the figure

$$\Delta CMO = \Delta CMP_2 + \Delta P_2MO,$$

$$rs \sin i = rs_2 \sin i + s_2 s \sin 2i,$$

$$\frac{1}{s_2} - \frac{1}{s} = \frac{2 \cos i}{r}. \quad . \quad . \quad . \quad 7.48$$

Let MP_1 be denoted by s_1 and let the various angles be described as in the figure.

Then

$$\Delta\theta = \frac{MM' \cos i}{s}, \quad \Delta\alpha = \frac{MM'}{r}, \quad \Delta\beta = \frac{MM' \cos i}{s_1}.$$

But

$$\beta = i + \alpha \quad \text{and} \quad 2i = \beta + \theta,$$

thus

$$\beta - \theta = 2\alpha,$$

and

$$\Delta\beta - \Delta\theta = 2\Delta\alpha.$$

Hence

$$\frac{1}{s_1} - \frac{1}{s} = \frac{2}{r \cos i}. \quad . \quad . \quad . \quad 7.49$$

The equations (7.48) and (7.49) give the distances of the two lines from the surface element.

$(s_2 - s_1)$ gives the astigmatic difference.

This distance depends upon the angle of incidence, and from the two equations we find

$$\frac{s_2 - s_1}{s_1 s_2} = \frac{2}{r} \sin i \tan i. \quad . \quad . \quad . \quad 7.50$$

It is clear that the astigmatic difference increases rapidly with the angle.

A similar treatment may be applied to the case of a concave reflecting surface. In this case the reflected rays actually pass through the two lines.

Astigmatism of a Refracting Spherical Surface

Another case is that of refraction at a convex spherical surface. We shall suppose that a point source is situated in air and that the refracting medium has a refractive index n.

If we refer to fig. 7.14, we see that the notation is similar to that of the previous figure. In this case we have $n \sin i = n' \sin i'$.

$$\Delta \mathrm{OMP}_2 = \Delta \mathrm{OMC} + \Delta \mathrm{CMP}_2,$$

$$ss_2 \sin(i - i') = sr' \sin i + s_2 r \sin i'.$$

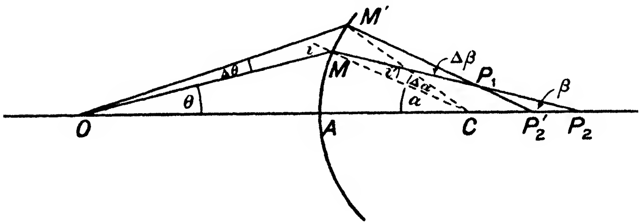

Fig. 7.14

It follows that

$$\frac{n'}{s_2} + \frac{n}{s} = \frac{1}{r}(n' \cos i' - n \cos i). \quad . \quad 7.51$$

This is the same formula as (7.12).

As in the example of reflection

$$\Delta\theta = \frac{\mathrm{MM}' \cos i}{s}, \quad \Delta\alpha = \frac{\mathrm{MM}'}{r}, \quad \Delta\beta = \frac{\mathrm{MM}' \cos i'}{s_1}.$$

We have also

$$\alpha = \beta + i', \quad i = \alpha + \theta.$$

From these equations together with $n \sin i = n' \sin i'$, we find

$$\Delta\alpha - \Delta\beta = \Delta i' = \frac{n \cos i}{n' \cos i'}(\Delta\alpha + \Delta\theta).$$

Hence

$$\frac{n\cos^2 i}{s} + \frac{n'\cos^2 i'}{s_1} = \frac{n'\cos i' - n\cos i}{r}. \quad 7.52$$

The equations (7.51) and (7.52) give s_1 and s_2, and the astigmatic difference may thus be obtained.

An alternative method of obtaining the astigmatic difference in this case is very useful in applications to successive surfaces.

In fig. 7.15 let O be a point source at distance y from the axis of the refracting surface. Suppose a small pencil fall from O on to the surface, and let OM_1 denote the principal ray. From the discussion above it is evident

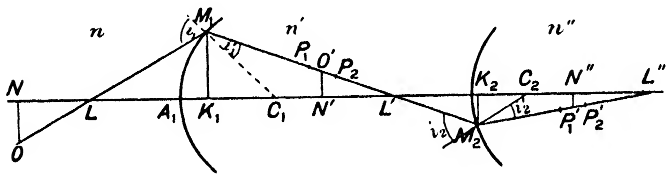

Fig. 7.15

that rays of the pencil in the plane of the figure meet at P_1 after refraction, and those meeting the surface on a circle through M_1 with centre on the line OC_1 intersect at P_2. P_1P_2 will be regarded as a small astigmatic difference, and if we draw a perpendicular from a point midway between these points to the axis, we can regard it as the image of ON. Let this perpendicular be of length y', while in the figure $ON = -y$.

It will be supposed that the angles of incidence and refraction are small enough to admit of the approximation

$$\cos i_1 = 1 - \tfrac{1}{2}i_1^2, \quad \cos^2 i_1 = 1 - i_1^2.$$

The points L and L′ are conjugate points on the axis, and we shall denote A_1L by l_1 and A_1L' by l_1'.

From (7.51) and (7.52)

$$\frac{n' \cos^2 i_1'}{\mathrm{M_1P_1}} + \frac{n \cos^2 i_1}{\mathrm{M_1O}} = \frac{n' \cos i_1' - n \cos i}{r}$$

$$= \frac{n'}{\mathrm{M_1P_2}} + \frac{n}{\mathrm{M_1O}}.$$

Thus

$$\frac{n'(1 - i_1'^2)}{\mathrm{M_1P_1}} + \frac{n(1 - i_1^2)}{\mathrm{M_1O}} = \frac{n'}{\mathrm{M_1P_1 + P_1P_2}} + \frac{n}{\mathrm{M_1O}}$$

$$= \frac{n'}{\mathrm{M_1P_1}}\left(1 - \frac{\mathrm{P_1P_2}}{\mathrm{M_1P_1}}\right) + \frac{n}{\mathrm{M_1O}},$$

whence

$$n' \frac{\mathrm{P_1P_2}}{s_1^2} = \frac{n' i_1'^2}{s_1} + \frac{n i_1^2}{s}.$$

But

$$n'i_1' = ni_1 \text{ and } i_1' = \angle \mathrm{M_1C_1K_1} - \mathrm{M_1L'K_1} = h_1\left(\frac{1}{r} - \frac{1}{l'}\right),$$

where $\mathrm{M_1K_1} = h_1$, and we are replacing angles by their tangents and thus neglecting again powers higher than the square. Thus

$$n' \frac{\mathrm{P_1P_2}}{s_1^2} = n'^2 h_1^2 \left(\frac{1}{r} - \frac{1}{l'}\right)^2 \left(\frac{1}{n's_1} + \frac{1}{ns}\right), \qquad 7.53$$

which gives the astigmatic difference $\mathrm{P_1P_2}$.

The Astigmatic Difference for a Lens

If the cone of rays after the first refraction passes on to strike a second spherical surface with the same axis, rays in the plane of the diagram meet in $\mathrm{P_1'}$ after refraction, and those from $\mathrm{P_2}$ cutting the surface on a circle through $\mathrm{M_2}$ with its centre on $\mathrm{P_2C_2}$ meet in $\mathrm{P_2'}$.

Then

$$\frac{n'' \cos^2 i_2'}{M_2P_1'} + \frac{n' \cos^2 i_2}{M_2P_1} = \frac{n''}{M_2P_2'} + \frac{n'}{M_2P_2}, \qquad 7{\cdot}54$$

$$\frac{n''(1 - i_2'^2)}{s_1'} + \frac{n'(1 - i_2^2)}{s_1}$$

$$= \frac{n''}{M_2P_1'}\left(1 - \frac{P_1'P_2'}{M_2P_1'}\right) + \frac{n'}{M_2P_1}\left(1 - \frac{P_1P_2}{M_2P_1}\right),$$

$$\frac{n''}{s_1'^2} P_1'P_2' + \frac{n'}{s_1^2} P_1P_2 = \frac{n'' i_2'^2}{s_1'} + \frac{n' i_2}{s_1}. \qquad 7{\cdot}55$$

But

$$n''i_2' = n'i_1 \quad \text{and} \quad i_2' = h_2\left(\frac{1}{l''} - \frac{1}{r_2}\right),$$

thus

$$\frac{n''}{s_1'^2} P_1'P_2' + \frac{n'}{s_1^2} P_1P_2$$

$$= n''^2 h_2^2 \left(\frac{1}{l''} - \frac{1}{r_2}\right)^2 \left(\frac{1}{n''s_1'} + \frac{1}{n's'}\right), \qquad 7{\cdot}56$$

where

$$s_1' = M_2P_1', \quad s' = M_2P_1 \quad \text{and} \quad -h_2 = M_2K_2.$$

By Helmholtz's relation (5·11)

$$ny\theta = -n'y'\theta'$$

or

$$ny \frac{h_1}{K_1N} = -n'y' \frac{h_1}{K_1N'}.$$

But we can write $K_1N = s$, $K_1N' = s_1$ to the necessary approximation, whence

$$\frac{ny}{s} = -\frac{n'y'}{s_1},$$

and similarly

$$\frac{n'y'}{s'} = -\frac{n''y''}{s_1'}.$$

Moreover,

$$\frac{O'N'}{O'L'} = \frac{M_1K_1}{M_1L} \quad \text{or} \quad \frac{y'}{l' - s_1} = \frac{h_1}{l'},$$

and similarly

$$\frac{h_2}{l''} = \frac{y''}{l'' - s_1'}.$$

Writing

$$\frac{n'y'}{s'} = -\frac{n''y''}{s_1'} = \mathrm{A},$$

we have

$$\frac{n''}{s_1'^2} = \frac{\mathrm{A}^2}{n''y''^2} \quad \text{and} \quad \frac{n'}{s'^2} = \frac{\mathrm{A}^2}{n'y'^2}.$$

Thus (7.56) becomes

$$\frac{P_1'P_2'}{n''y''^2} + \frac{P_1P_2}{n'y'^2} = \frac{n''^2h_2^2}{\mathrm{A}^2}\left(\frac{1}{l''} - \frac{1}{r_2}\right)^2\left(\frac{1}{n''s_1'} + \frac{1}{n's'}\right).$$

But

$$n''h_2 = \frac{n''y''l''}{l'' - s_1'} = \frac{-\mathrm{A}s_1'l''}{l'' - s_1'},$$

thus the last equation becomes

$$\frac{P_1'P_2'}{n''y''^2} + \frac{P_1P_2}{n'y'^2} = \frac{\left(\frac{1}{l''} - \frac{1}{r_2}\right)^2}{\left(\frac{1}{l''} - \frac{1}{s_1'}\right)^2}\left(\frac{1}{n''s_1'} + \frac{1}{n's'}\right). \qquad 7.57$$

Equation (7.53) becomes in the same way

$$\frac{P_1P_2}{n'y'^2} = \frac{s_1^2h_1^2}{y'^2}\left(\frac{1}{l'} - \frac{1}{r_1}\right)^2\left(\frac{1}{n's_1} + \frac{1}{ns}\right)$$

$$= \frac{\left(\frac{1}{l'} - \frac{1}{r_1}\right)^2}{\left(\frac{1}{l'} - \frac{1}{s_1}\right)^2}\left(\frac{1}{n's_1} + \frac{1}{ns}\right). \quad . \quad . \quad 7.58$$

The similarity of the squared factors in (7.57) and (7.58) should be noticed. We can represent them by Q''^2 and Q'^2 respectively. Subtracting these equations, we obtain

$$\frac{P_1'P_2'}{n''y''^2} = Q''^2\left(\frac{1}{n''s_1'} + \frac{1}{n's'}\right) - Q'^2\left(\frac{1}{n's_1} + \frac{1}{ns}\right). \quad 7.59$$

This gives the astigmatic difference $P_1'P_2'$ for the lens, and the condition for the absence of astigmatism is the vanishing of the right-hand side.

It should be noted that we have tacitly assumed the presence of a narrow stop at L′ to keep the pencil incident on the second surface limited so that it falls over a small area.

Astigmatism Considered more Generally

These examples serve to illustrate astigmatism and to introduce the astigmatic difference, which is an important quantity in connection with image formation. It is clear that similar considerations may be applied to more complex systems and that there will in these cases be two focal lines.

In order to examine this property of optical systems in a more general way, we shall require two theorems of fundamental importance in optics. We take this opportunity of introducing them although our use of them is limited.

(i) *Fermat's Principle of Least Time*

This principle states that, if a ray of light passes from one point to another, undergoing reflections and refractions, the path taken is such that the time occupied over it is stationary, i.e. the difference in the time taken over the actual path and that taken over an infinitely near path beginning and ending at the same two points is infinitesimal. The value may be a maximum, a minimum or neither, but in any case it will be stationary.

If the simple case of reflection at a plane mirror be considered, the ray from a point A passing after reflection through a point B, it will be seen that the path in this case is a minimum.

It can be proved that this principle is true provided the laws of reflection and refraction be assumed, and, conversely, if the principle is true these laws may be demonstrated.

The principle may be accepted as the starting-point in the theory of rays of light, occupying a fundamental place in this theory like that of Hamilton's Principle or the Principle of Least Action in Mechanics. The similarity of these stationary principles, first revealed by Hamilton, has only come into prominence in recent years through the work of de Broglie and of Schroedinger in the realm of Atomic Physics. The principle can be expressed in another form. Let ds denote an element of the path and V the velocity at this element. The element of time corresponding to this is $\frac{ds}{V}$ or $\frac{nds}{V_0}$, where n is the refractive index of the medium in which ds lies and V_0 is the velocity in vacuo.

Thus the law is $\int\frac{nds}{V_0}$ or $\int nds$ is stationary. $\int nds$ is called the reduced path, so that the principle becomes that of the stationary reduced path.

(ii) *The Law of Malus*

Fermat's principle can be applied to show that a system of rays originally normal to a surface will still be normal to a surface after reflections and refractions have taken place.

Let PP′ (fig. 7.16) denote a surface to which the rays are originally normal. Let two adjacent rays PQRS, P′Q′R′S′ undergo reflections and refractions. In the figure QQ′ represents a refracting surface, RR′ a reflecting surface.

Let PQRS, P′Q′R′S′ be two distances for which

$\int nds$ is the same. By Fermat's Principle the adjacent paths PQRS and PQ′R′S have the same value for the reduced path, or more strictly, the difference in these reduced paths is an infinitesimal quantity.

But since P′Q′ is normal to PP′, the difference between Q′P and Q′P′ is infinitesimal.

Taking away the common parts of PQ′R′S and P′Q′R′S′, we then have SR′ = R′S′. Thus these lines are perpendicular to SS′. This is true for every ray adjacent to RS, consequently a surface can be drawn through the ends of these rays which is perpendicular to them all. This is known as the law of Malus.

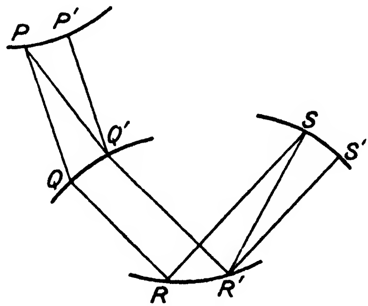

Fig. 7.16

Astigmatic Pencils

Rays from a point are normal to a sphere drawn about the point as centre. Thus after emerging from an optical system the rays are normal to a surface not, of course, necessarily a sphere. But the normals to a surface do not necessarily all pass through a point, so that the emergent pencil is, in general, astigmatic.

The Focal Lines

The pencils we consider are described as thin pencils, which means that they consist of a ray known as the principal ray and of other rays not far from it. The inclinations of these rays to the principal ray are small quantities of the first order.

Let a thin astigmatic pencil cut its orthogonal surface in a small area dS. We know from the geometry of surfaces that two systems of curves can be drawn on a

surface known as lines of curvature. These lines cut each other at right angles and can be regarded as dividing the surface into small curvilinear rectangles.

Let us draw the rectangle that encloses dS, (ABCD in fig. 7.17). Let the normals to AB at the ends of the element of the curve meet at *ab*. The curvature of the element CD parallel to AB will differ only very slightly from that of AB, so that if the corresponding normals to this element meet at *cd*, the line F_1, which joins *ab* to *cd*, will run parallel to the surface element and will be very approximately a straight line parallel to AD or BC.

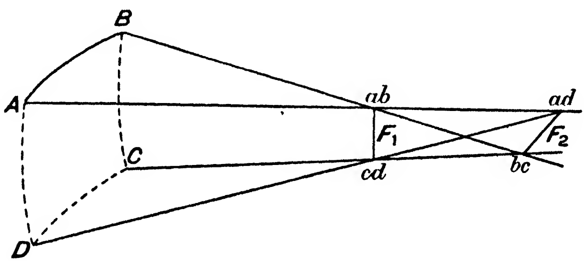

Fig. 7.17

In the same way the normals to the elements BC and AD give rise to the line F_2.

F_1 and F_2 are perpendicular to one another, and in general are situated at different distances from the element because the curvatures of the two sets of lines of curvature are in general different. They are described as the focal lines of the pencil, and their separation is known as the astigmatic difference.

In the special cases when the two curvatures are the same, these two lines become a single point, and we have true image formation. In general no such image exists. If a screen be moved from the position of one line to that of the other, the illuminated area will widen in a direction perpendicular to one line and become narrower in the direction of this line, finally becoming the other line.

At some intermediate position the area will be of equal width in both directions and, under certain circumstances, it will thus be circular. This circular area is called the circle of least confusion and is the closest approximation to an image in astigmatic systems.

Graphical Representation of Astigmatism

Our discussion shows that in general a small oblique pencil incident upon an optical system gives rise to an astigmatic pencil in the image space. In fig. 7.18 let AB denote the principal ray of the pencil in the image space. The two points L_1 and L_2 denote the positions of the focal lines.

Corresponding to different inclinations of the incident ray, AB will have different inclinations and L_1 and L_2 different positions. The astigmatic difference will have different values.

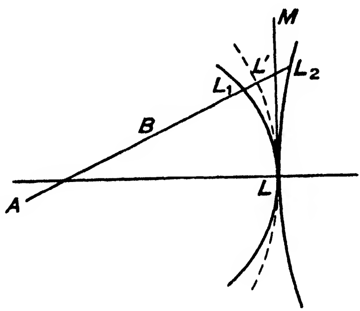

Fig. 7.18

In the plane of the figure L_1 and L_2 lie on two curves which touch at L, the point corresponding to the paraxial zone where the astigmatism vanishes.

Actually in three dimensions, the focal lines lie on two surfaces touching at L, but we are considering only a section of this surface.

If we draw a line LM through L at right angles to the optic axis, we can plot a curve with axes LM and the optic axis showing the relation of the displacements of L_1 and L_2 from these lines. It is possible by the use of a stop with an optical system to limit the inclinations of the pencils and to modify the astigmatic difference. When this is reduced to zero, L_1 and L_2 unite in a point L' which lies on another surface, and the system is said

to be astigmatically corrected. But L′ is not necessarily on a plane perpendicular to the axis. If this is the case, the system is said to be anastigmatically flattened. If it is impossible to do this, it is at least possible to bring the circle of least confusion on to a plane LM. This is brought about in the case of a lens by placing a stop at the proper distance in front of it. The effect of this stop is to cut out from each oblique pencil a small partial pencil, and since the stop is at some distance from the lens, the smaller pencils at different inclinations traverse different parts of the lens, the most oblique traversing the peripheral region. It is important that the stop should not lie close to the lens, otherwise all the pencils traverse approximately the same region, and the effect would be lost. It is due to the fact that the pencils at different inclinations are made to pass through well-separated parts of the lens that it is possible to displace the circles of least confusion to lie in a plane. This process may be described as artificial flattening of the image to distinguish it from true flattening, which refers to the process of making the image plane by purely optical methods without the use of a stop. Artificial flattening is obtained at the expense of brightness, since the pencils are cut down by the stop.

The process is illustrated in the separation of lenses as a means of flattening the image. When two lenses are together, it is easy to see that the image will be curved, for they act as a simple lens, although there is some improvement in the image on account of the division of labour between the two systems. When the lenses are separated, the first acts as a stop, cutting off part of the oblique pencils, but here again there is a loss of brightness.

III. Curvature of Field

Spherical aberration and astigmatism are two examples of aberrations which cause a departure from the collinear

relation between object and image which is characteristic of the paraxial zone. We now consider another of these aberrations described as curvature of field. In order to illustrate what is meant by this term, we can examine it in connection with a concave reflecting surface.

In fig. 7.19 let OM denote an extended object with its foot O on the principal axis of the surface AA′ of which the centre of curvature is situated at C. An image of O will be situated at a point O′ on CO, but unless we limit ourselves to thin pencils we shall have the effects of spherical aberration, and unless the pencils fall centrically on the mirror we shall have astigmatism.

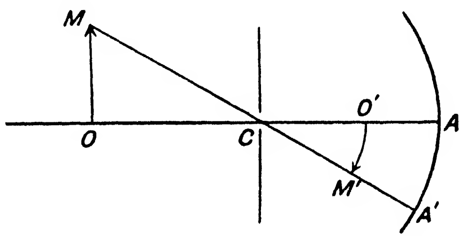

Fig. 7.19

Both these effects can be avoided by placing a stop at C with a small hole about that point, through which thin centrical pencils can fall on the mirror from all points of the object.

It is at once clear that the image formed will not be linear, for O, O′ and M, M′ are pairs of conjugate points lying on the radii OCA and MCA′. But $CO/CO' \neq MC/CM'$.

Thus even in the absence of spherical aberration and astigmatism there is this further aberration. Although the image O′M′ is sharp, it is not flat, and a screen placed at O′ parallel to the object would be blurred away from the centre.

In any complex optical system which consists of a combination of reflecting or refracting surfaces, we have

to consider the successive effects of the reflections and refractions on the curvature of the image.

We shall illustrate the theory of this by examining the case of successive refractions at spherical surfaces.

Curvature for a Single Refraction at a Spherical Surface

Let AB (fig. 7.20) denote the spherical boundary between two media of indices n and n'. Let OP denote a curved object and O′P′ the corresponding image. Let the radius of curvature of OP be R and let R′ be that of O′P′. These radii will be considered positive if the centres of curvature be to the right of O and O′ respec-

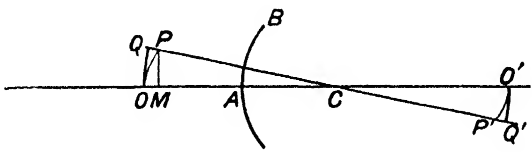

Fig. 7.20

tively, so that the sign convention will agree with that previously introduced. Let r denote the radius of curvature of the surface AB.

Let OQ be an arc of a circle of radius of curvature CO, and let O′Q′ be a similar arc of radius CO′. Thus OQ and O′Q′ will be in the relation of object and image with respect to the refracting surface.

Draw the line QCQ′ cutting the arcs in Q, P and Q′, P′ respectively. In estimating the curvature near O and O′ it is necessary to suppose that P lies near O and similarly P′ lies near O′. Thus we shall regard both P and Q as situated at a distance h from the axis and likewise P′ and Q′ as at a distance h'.

Imagine perpendiculars drawn from P and Q to the axis and similarly from P′ and Q′. Then we have to the approximation necessary

$$OM \,.\, 2R = h^2$$

or

$$\mathrm{OM} = \frac{h^2}{2\mathrm{R}}.$$

Similarly, if N is the foot of the perpendicular from Q,

$$\mathrm{ON} = \frac{h^2}{2(l + r)},$$

where $l = \mathrm{AO}$. Thus

$$\mathrm{NM} = \frac{h^2}{2}\left(\frac{1}{\mathrm{R}} - \frac{1}{l + r}\right).$$

In the same way $\mathrm{N'M'} = \dfrac{h'^2}{2}\left(-\dfrac{1}{\mathrm{R'}} - \dfrac{1}{l' - r}\right)$, remembering that the curvature of O′P′ as drawn in the figure is negative.

From (3.19) we know that the relation between the distances AO, AO′ and AM, AM′ is given by

$$\frac{n'}{l'} + \frac{n}{l} = \frac{n' - n}{r}. \qquad \ldots \quad 3.19$$

Thus by differentiation

$$\frac{n'}{l'^2}\Delta l' + \frac{n}{l^2}\Delta l = 0$$

or

$$\frac{\Delta l'}{\Delta l} = -\frac{n}{n'}\frac{l'^2}{l^2},$$

$$\frac{\mathrm{M'N'}}{\mathrm{MN}} = -\frac{n}{n'}\frac{l'^2}{l^2}. \qquad \ldots \quad 7.60$$

If we remember that PA and AP′ are conjugate rays and that the angles PAO and P′AO′ are small, we have

$$n \sin \mathrm{PAO} = n' \sin \mathrm{P'AO'},$$

$$n\frac{h}{l} = n'\frac{h'}{l'},$$

whence by (7.60)

$$\frac{M'N'}{MN} = -\frac{n'}{n}\left(\frac{h'}{h}\right)^2.$$

This relation combined with the values obtained for NM and N′M′ gives

$$\frac{1}{R'} + \frac{1}{l'-r} = \frac{n'}{n}\left(\frac{1}{R} - \frac{1}{l+r}\right),$$

$$\frac{1}{n'R'} - \frac{1}{nR} = -\left(\frac{1}{n(l+r)} + \frac{1}{n'(l-r)}\right).$$

By means of (3.19) this may be put into the form

$$\frac{1}{n'R'} - \frac{1}{nR} = \frac{n-n'}{nn'}\cdot\frac{1}{r}. \qquad . \quad . \quad . \quad 7.61$$

This gives us the relation between the curvature of an object and that of its image or, as it may be stated, the relation between the curvature of the object and image spaces.

The Relation between the Curvatures after Successive Refractions

If the light is refracted at a second surface of radius r_1 into a medium of refractive index n'', we have for the second refraction

$$\frac{1}{n''R''} - \frac{1}{n'R'} = \frac{n'-n''}{n'n''}\cdot\frac{1}{r_1},$$

On combining this with (7.61) we obtain

$$\frac{1}{n''R''} - \frac{1}{nR} = \frac{n-n'}{nn'}\cdot\frac{1}{r} + \frac{n'-n''}{n'n''}\cdot\frac{1}{r_1}. \qquad 7.62$$

The process may be extended to any number of surfaces.

If we apply the formula to the case of a thin lens in air,

i.e. with $n = n'' = 1$, and assuming that the object is linear, i.e. $R = \infty$, we find

$$\frac{1}{R''} = \frac{1 - n'}{n'} \cdot \frac{1}{r} + \frac{n' - 1}{n'} \cdot \frac{1}{r_1} = \frac{n' - 1}{n'} \cdot \left(\frac{1}{r_1} - \frac{1}{r}\right)$$

$$= -\frac{1}{n'f}, \qquad \ldots \qquad 7.63$$

where f is the focal length of the lens.

For a series of thin lenses

$$\frac{1}{R'} = -\Sigma \frac{1}{n'f}, \qquad \ldots \qquad 7.64$$

where R′ denotes the curvature of the image finally obtained. If the right-hand side of this equation vanishes, the image will be flat in the neighbourhood of the axis.

Distortion

We saw in connection with equations (7.8) and (7.14) that the different zones of an optical system have different powers, and we shall thus expect that the image formed when wide-angled pencils contribute to it will not be uniformly magnified in all its parts.

We have just seen that the image is not generally flat, and this can also be appreciated from fig. 7.18 and from the remarks on astigmatism made in connection with this figure. But we are usually interested in the picture formed on a flat surface, as on LM of the figure. If the system is not anastigmatically flat, the picture will consist of " blur " circles in the plane, and this will approximate to an image if the circles of least confusion have been caused to lie on a plane by the use of a stop. We shall assume that the pictures we are discussing here are flat in this sense and shall regard them as images.

To illustrate this case we shall consider image formation under special circumstances of an extended flat object by means of a convex lens.

Let the extended flat object be OP_3 (fig. 7.21(a)), and let there be a small stop situated in front of the lens at B. Suppose that the small central pencil from O which passes through the stop is focussed at O′. We assume that a plane image is formed on $O'P_3'$. A small oblique pencil from P_1 is represented in the figure by the principal ray P_1B of the pencil. After refraction it will cut

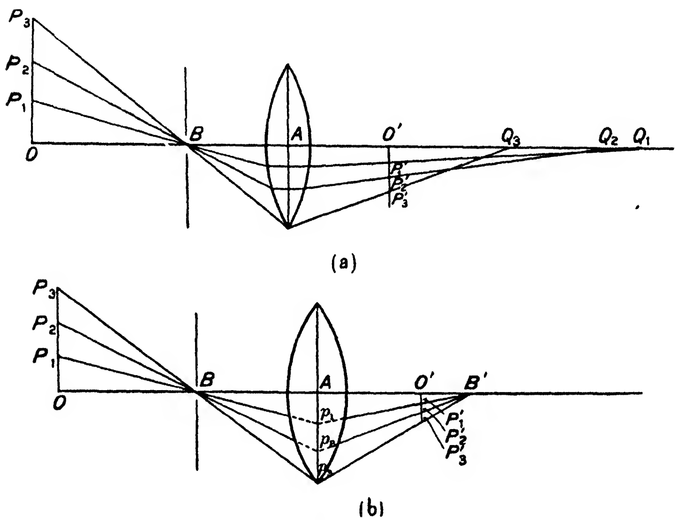

Fig. 7.21

the axis at a point beyond O′. The same is true for other pencils from P_2, P_3, but the principal rays cut the axis at points nearer and nearer to O′.

Our assumption about the formation of a flat image means that the image points O', P_1', P_2', P_3' all lie on a plane. If the image is to be without distortion, then when $OP_1 = P_1P_2 = P_1P_3$, it is necessary that $O'P_1' = P_1'P_2' = P_2'P_3'$. This requires that all the rays, which intersect in the object space at B, should intersect at a point in the image space. This means that B should be an aplanatic point

for the lens; the centre of the stop must be at an aplanatic point. All the points Q_1, Q_2, Q_3 will then coincide in a single point B′ (fig. 7.21(*b*)). A further requirement is that the incident and emergent rays should intersect in points p_1, p_2, p_3 which lie in a plane perpendicular to the axis. This is shown in the lower figure. This condition must be satisfied for exact similarity between object and image. The system is described as orthoscopic when the condition is fulfilled. When exact similarity does not exist, we may get $O'P_1' > P_1'P_2' > P_2'P_3'$, or the sign of inequality may be reversed. If a rectangular network of lines forms the object, the character of the images in the two cases is shown in fig. 7.22. The former case gives " barrel-shaped " distortion, the latter gives " pin-cushion distortion. "

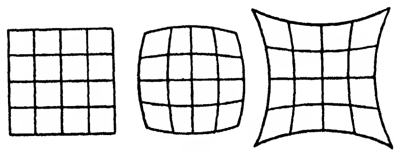

Fig. 7.22

If θ and θ' are the inclinations of two corresponding principal rays in fig. 7.21(*b*), it is clear that the ratio $\dfrac{\tan\theta'}{\tan\theta}$ is the same for all pairs of rays when the orthoscopic condition is satisfied.

The Orthoscopic Condition

In this section we shall consider further the question of orthoscopy.

In fig. 7.23 let B and B′ denote aplanatic points and let OP_3 be an extended plane object at right angles to the optic axis. The lines drawn to B denote the principal rays of narrow pencils, and those from B′ the corresponding principal rays of the image pencils. We may suppose that the image $O'P_3'$ has been anastigmatically flattened, but if not, it will be made up of blur patches in this plane, and the continuous and dotted curves of the figure

will then denote the astigmatic curves. In practice such image flattening is possible, at any rate approximately, e.g. we see from (7.64) that the greatest flatness in the use of two lenses results when

$$n_1 f_1 + n_2 f_1 = 0, \quad . \quad . \quad . \quad . \quad 7.65$$

where f_1 and f_2 denote the focal lengths of the lenses and n_1 and n_2 the refractive indices of the materials of which

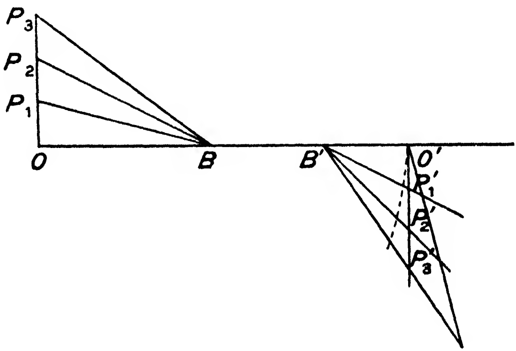

Fig. 7.23

they are made. In this case if the object curvature is zero, the image curvature near the axis is also zero provided that $\Sigma \frac{1}{n'f}$ vanishes, i.e. if

$$\frac{1}{n_1 f_1} + \frac{1}{n_2 f_2} = \frac{n_1 f_1 + n_2 f_2}{n_1 n_2 f_1 f_2} = 0.$$

We may thus assume that OP_3 and $O'P_3'$ are conjugate planes. If the object and image are geometrically similar,

$$\frac{OP_1}{OB} = \frac{O'P_1'}{O'B'}, \quad \frac{OP_2}{OB} = \frac{O'P_2'}{O'B'}.$$

Since B and B′ are conjugate points, they can be taken as origins for the two spaces respectively, and we measure

distances from them and the angles of inclination in accordance with our notation.

Consider the ray BP_2 and its conjugate $B'P_2'$. Let the co-ordinates of P_2 be denoted by (l, y) and of P_2' by $(l'\ y')$, where

$$l = BO, \quad l' = B'O',$$

$$y = OP_2, \quad y' = -O'P_2'.$$

The angles of inclination for the ray P_2B and its conjugate $P_2'B'$ are θ and θ', where

$$\theta = \angle P_2BO, \quad \theta' = -P_2'B'O'.$$

Thus

$$\tan\theta = \frac{P_2O}{OB} = \frac{y}{l},$$

$$\tan\theta' = -\frac{P_2'O'}{O'B'} = \frac{y'}{l'},$$

whence

$$\frac{\tan\theta'}{\tan\theta} = \frac{y'}{y}\cdot\frac{l}{l'}. \qquad . \quad . \quad . \quad . \quad . \quad . \qquad 7.66$$

For the pair of aplanatic points $\frac{l'}{l}$ is constant, which simply means that all the rays in the object space go through B and those in the image space through B'. The orthoscopic condition requires that $\frac{y'}{y}$ should be constant. Thus the condition is expressed by the tangent relation

$$\frac{\tan\theta'}{\tan\theta} = \text{constant} \qquad . \quad . \quad . \quad . \quad . \qquad 7.67$$

for all the conjugate pairs of rays.

We can modify this result to make it applicable to the case where spherical aberration is not exactly eliminated.

This means that rays outside the paraxial region do not go through B′ exactly. Suppose that the paraxial value of l' is denoted by l_0' and that δ' denotes the spherical aberration for a particular ray. Thus the distance B′O′ or l_0' must be replaced by $(l_0' + \delta')$, where δ' depends on θ'. We can make the result more general by considering the possibility that all the rays do not pass through B exactly and replace l by $(l_0 + \delta)$ in this case, δ varying with θ. The equation (7.66) becomes

$$\frac{\tan\theta'}{\tan\theta} = \frac{y'}{y} \cdot \frac{l_0 + \delta}{l_0' + \delta'},$$

and since $\frac{y'}{y}$ is a constant for an orthoscopic system, we can write

$$\frac{\tan\theta'}{\tan\theta} = k \cdot \frac{l_0 + \delta}{l_0' + \delta'} \qquad . \quad . \quad . \quad 7.68$$

as the orthoscopic condition.

The case where the object is situated at a great distance is a very important one in photographic work, and the condition obtained is fundamental in connection with photographic objectives.

Let X denote the distance of the object from the principal focus, and let X_B denote the corresponding distance of the point where the ray cuts the axis, i.e. the distance of the point B or of a point close to it. Then

$$X - X_B = l_0 + \delta.$$

From (6.21) we have

$$k = \frac{y'}{y} = -\frac{f}{X}. \qquad . \quad . \quad . \quad 6.21$$

Thus

$$(l_0 + \delta)k = -f + kX_B.$$

For a distant object k becomes very small, so that $(l_0 + \delta)k$ becomes equal to f.

In this case (7.68) becomes

$$\frac{\tan\theta'}{\tan\theta} = \frac{-f}{l_0' + \delta'}. \quad . \quad . \quad . \quad . \quad 7.69$$

Coma and the Sine Condition

In this chapter we have considered four causes of deviation from the collinear relationships when we leave the paraxial zone of optical systems. These deviations are all known as aberrations, and they are spherical aberration, astigmatism, curvature of field and distortion. Another aberration is known as coma, and it is usually placed second in this series.

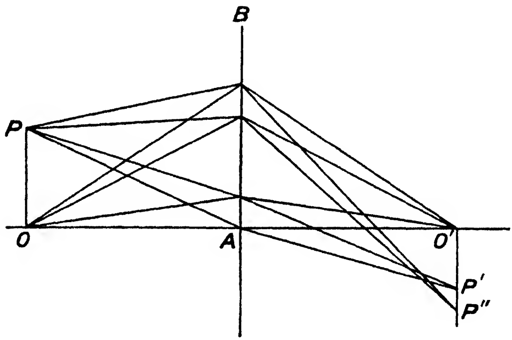

Fig. 7.24

In order to understand the meaning of this, let us suppose that an optical system (fig. 7.24) exhibits no spherical aberration for the point O. Thus all rays from O, however much they are inclined to the optic axis OA, are brought to a focus at O′. These conjugate points are aplanatic points. Now consider rays from a point P on a plane surface through O. The question arises: will all the rays from this point be focussed at a point on a conjugate surface? The fact that O and O′ are aplanatic points is no guarantee that this desired result will follow. Usually a blur of light occurs on one side of O′ which is known as coma.

A special condition must be satisfied to avoid its occurrence.

We shall now investigate the condition to be satisfied in order that a *small* plane object at O will give rise to a sharp image at O′. According to Abbe, when this condition is attained, it can be explained by the fact that for the small element of surface at O the different zones of the system have the same magnifications. This is illustrated in fig. 7.24. When the system is corrected to satisfy the condition, all the pencils from P are brought into coincidence at one point P′, not, as the diagram shows, some at P′ and others at P″.

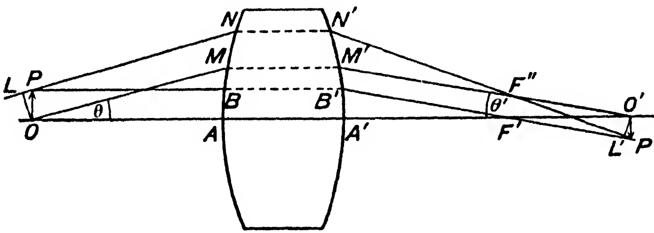

Fig. 7.25

We shall see that the condition to be satisfied is that the sines of the angles made with the axis by all the rays from O bear a constant ratio to the sines of the angles made with the axis by the conjugate rays through O′. In writing on this subject Abbe uses the word aplanatic for the points O and O′ only when this condition is fulfilled. We have used the word, in common with other writers, in the less restricted sense.

We shall suppose that the system consists of a thick lens for the purpose of illustration, but the discussion applies to optical systems in general.

We shall use the method of Hockin, which is that generally adopted in deriving the relation. The basis of the method is Fermat's Principle of Extreme Path.

Let AN (fig. 7.25) be the boundary of the system on

the side of the incident light and A′N′ be the boundary on the opposite side. Let OP denote a small object at right angles to the axis, O being an aplanatic point for the system. Thus rays inclined to the axis OA, even at large angles, are brought to a focus at the second aplanatic point O′.

We are investigating the condition that an image shall be formed at O′P′.

Let [OAA′O′] denote the optical path $\int n\,ds$ for the ray OAA′O′. By Fermat's principle the optical path taken by a ray from a point to the conjugate point has a stationary value. Thus

$$[OAA'O'] = [OMM'O'] \quad \text{(see fig. 7.25)}.$$

From P draw a ray PB parallel to the axis and a ray PN parallel to OM. Then

$$[PBB'P'] = [PNN'P'],$$

since P and P′ are conjugate points.

The rays PB and OA may be regarded as arising from an object at infinity. Thus the optical paths of these rays from infinity to F′ are equal, and since OP is perpendicular to the parallel rays

$$[PBB'F'] = [OAA'F'],$$

O′P′ is assumed to be small, i.e. the angle O′F′P′ is a small quantity of the first order, and consequently O′F′ = P′F′ to this order of magnitude. Thus

$$[PBB'P'] = [OAA'O'],$$

whence

$$[PNN'P'] = [OAA'O'] = [OMM'O'].$$

Drop a perpendicular OL to PN and O′L′ to P′N′. Since F″ is conjugate to a point at infinite distance, i.e. to the point from which the rays PN and OM may be supposed to arise,

$$[LNN'F''] = [OMM'F''].$$

If this is combined with

$$[PNN'P'] = [OMM'O'],$$

we have from their difference

$$[F''P'] - [F''O'] = [LP]. \quad . \quad . \quad 7{\cdot}70$$

But

$$[F''P'] - [F''O'] = n'(F''P' - F''O'),$$

where n' is the refractive index of the region in which these rays are situated, i.e. of the image space.

Since $O'P'$ is small,

$$F''P' - F''O' = P'L' = O'P' \sin\theta'.$$

In the same way

$$LP = OP \sin\theta.$$

Thus from (7.70)

$$n'O'P' \sin\theta' = nOP \sin\theta$$

or

$$\frac{n \sin\theta}{n' \sin\theta'} = \frac{O'P'}{OP} = \text{linear magnification}, \quad 7{\cdot}71$$

where θ and θ' are the angles marked in the diagram.

If the ratio of the sines is constant, the linear magnification is constant for all zones. All the rays from P, whatever their inclination to the axis, are focussed at P′, and we have a sharp small image near O′. It should be noted that an extended surface is not necessarily sharply delineated. For from (6.25) we see that the collinear relation requires that for an object the ratio of the tangents of θ and θ' should be constant. This again is the requirement of the orthoscopic condition. A point to point imaging of extended objects by wide beams is impossible; it is only when the angles θ and θ' are small that the tangent and sine conditions are compatible. Another important point is that a system can be aplanatic in Abbe's sense of the term for one position

of the object only. Sharp delineation for elements one behind another is thus impossible.

In order to prove this, it is necessary to show that the sine law cannot be fulfilled for two different points on the axis.

In fig. 7.26 let O and O_1 denote two point sources of light and, if possible, suppose that wide angle beams are focussed by the system at O′ and O_1' respectively. Draw two parallel rays O_1N and OM at an inclination θ. Then, using the same notation as before,

$$[OMM'O'] = [OAA'O']$$
$$[O_1NN'O_1'] = [O_1AA'O_1'].$$

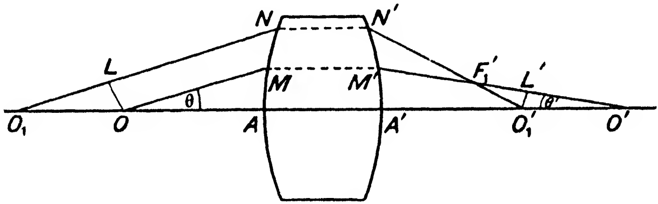

Fig. 7.26

Let OL and $O_1'L'$ be drawn perpendicular to O_1N and $O'M'$ respectively. Thus

$$[LNN'O_1'] = [OMM'L'],$$

and consequently

$$[O_1NN'O_1'] - [OMM'O'] = [O_1L] - [L'O']$$
$$= nO_1L - n'O'L'.$$

But

$$[O_1AA'O_1'] - [OAA'O'] = [O_1O] - [O_1'O']$$
$$= nO_1O - n'O_1'O',$$

the same approximations being made as before. Thus

$$n(O_1O - O_1L) = n'(O_1'O' - O'L'),$$

i.e.

$$nO_1O(1 - \cos\theta) = n'O_1'O'(1 - \cos\theta')$$

or

$$\frac{n\sin^2\frac{1}{2}\theta}{n'\sin^2\frac{1}{2}\theta'} = \frac{O_1'O'}{O_1O}. \quad . \quad . \quad . \quad 7{\cdot}72$$

Thus for the existence of two aplanatic points we have to satisfy the condition $\dfrac{\sin\frac{1}{2}\theta}{\sin\frac{1}{2}\theta'} = \text{constant}$, and this is not compatible with (7·71). If O and O_1 are two points close together on the axis, we see that (7·72) is the condition for the imaging of an element on the axis.

CHAPTER VIII

THE ABERRATIONS FROM THE POINT OF VIEW OF THE WAVE THEORY OF LIGHT

It is important to remember that the rays of light which we have been considering in this geometrical treatment of optics are pure abstractions. The fact that useful results can be deduced by this method of studying the subject means that we can regard the light emitted from a small source as made up of narrow pencils or rays which on reflection or refraction obey certain geometrical laws. But the study of light in Physics shows that we must consider light as composed of waves if we wish to describe the observed phenomena accurately. If we make use of the wave aspect of the subject, we connect it with physical optics, and then geometrical optics appears as a part of the whole. It is that part of the whole subject of Optics in which it is sufficient to consider light as if it travelled along straight lines.

Although the treatment of the subject by means of waves is strictly outside the scope of a work on geometrical optics, it is impossible to grasp the subject properly without turning our attention briefly to the study of light waves. In particular it is indispensable in the study of optical instruments.

We can learn something of the properties of waves by considering those which may be seen to travel over the surface of water. Suppose that we have a sheet of water of uniform depth. This provides us with a homogeneous medium for the waves to travel in, a difference between these water waves and light waves being that we observe

the former travelling in the water surface, i.e. as two dimensional waves, while the latter travel in three dimensions. If a stone be dropped into the water, circular waves travel outward from a centre. The velocity of the waves is the same in all directions, and they preserve their circular form. In the case of light waves the velocity in a uniform medium such as empty space, air, or in glass is also the same in all directions, but the waves travel outward from a small source as spheres with the source as centre.

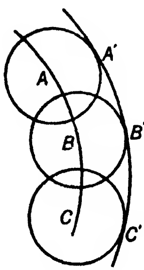

Fig. 8.1

If the depth of the water is different in different directions, the waves on its surface are no longer circular because of the different velocities in different directions. A similar change occurs in light waves when the medium is no longer homogeneous.

If we observe a small piece of cork lying in the surface of the water, it will be noted that, as the waves pass it, it rises and falls and becomes itself the origin of waves spreading out from it as a centre. This observation helps us to understand a very important principle applied to wave propagation.

As the wave travels forward we recognize a " front " of the wave on which the particles of the medium are in the same state of motion. We describe them as being in the same phase. Each of these particles, according to Huyghen's principle, behaves like the cork and is the origin of wavelets. These wavelets travel outward, and the new wave front is the envelope of the wavelets.

Let ABC in fig. 8.1 denote a wave front. Wavelets are to be imagined as centred at the points of this wave front. The new wave front at a time t seconds later is A′B′C′, a surface which touches the wavelets.

In the figure ABC is drawn as an arc of a circle to represent a spherical wave. The wavelets drawn with centres A, B and C spread out to become spheres of

equal radii after an interval of time, and the envelope of the wavelets is a larger sphere concentric with ABC. A ray passing through A is obtained by joining A to the point of contact of the wavelet with the envelope. In the example this is A′, and AA′ is along the radius to the spherical wave fronts, so that in this case the ray is at right angles to the wave front. This is not always the case, for if the wave does not travel outward as a spherical wave the ray AA′ is not necessarily normal to the wave front.

A simple wave is represented by the equation

$$y = a \sin 2\pi\nu\left(t - \frac{r}{v}\right). \quad . \quad . \quad . \quad 8.1$$

We can think of this as representing a displacement y at a time t at distance r from an origin (fig. 8.2). If we fix our attention upon any point and examine the variation of y at this point as time elapses, we see that we have an oscillation of y with frequency ν. If we fix our attention on a particular instant and study the value of y at different distances r, we note that y varies along r from zero to a maximum, back to zero and to a minimum, and so on, the distance between successive maxima being $\lambda = v/\nu$. Further, a condition at a point r_1 at time t_1 will exist at r_2 at time t_2 if

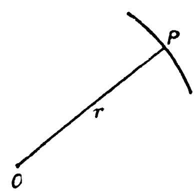

Fig. 8.2

$$t_2 - r_2/v = t_1 - r_1/v,$$

i.e. if

$$v = \frac{r_2 - r_1}{t_2 - t_1}.$$

Thus it appears as if the condition travels from r_1 to r_2 with velocity v. We speak of this velocity as the phase velocity. Its value differs for different media, and the ratio of the velocity in a medium to that in a vacuum is called the refractive index.

If at time t_1 at distance r_1 the value of $2\pi\nu\left(t - \frac{r}{v}\right)$ is the same as at time t_2 at distance r_2, the conditions at the two points are said to be in the same phase. The expression $2\pi\nu\left(t - \frac{r}{v}\right)$ is a measure of the phase.

We are often concerned with differences in phase at a point in the case of waves arriving by different routes to the point. The importance of this is that the effects of two light waves arriving at the same point are additive. To put this more precisely, suppose that a wave $y_1 = a_1 \sin 2\pi\nu\left(t - \frac{r_1}{v}\right)$ and a wave $y_2 = a_1 \sin 2\pi\nu\left(t - \frac{r_2}{v}\right)$ arrive at a particular point at time t after having travelled distances r_1 and r_2 respectively from two identical sources. The additive principle, known as the principle of superposition, states that the effect at the point is represented by a displacement y equal to the sum of y_1 and y_2,

$$y = a_1\left[\sin 2\pi\nu\left(t - \frac{r_1}{v}\right) + \sin 2\pi\nu\left(t - \frac{r_2}{v}\right)\right].$$

The amplitudes a_1, frequencies ν, and velocities v are taken equal for simplicity.

It follows that

$$y = 2a_1 \sin 2\pi\nu\left(t - \frac{r_1 + r_2}{2v}\right) \cos 2\pi \frac{\nu}{v}\left(\frac{r_1 - r_2}{2}\right) \qquad 8.2$$

and that the value of y depends upon $(r_1 - r_2)$. It is zero if $(r_1 - r_2)$ is an odd number of half wave-lengths, i.e. if $r_1 - r_2 = (m + \frac{1}{2})\frac{v}{\nu} = (m + \frac{1}{2})\lambda$ and has a maximum value if $(r_1 - r_2) = m\lambda$, m denoting an integer. The intensity of the light at any point depends upon the square of the amplitude of the displacement, i.e. upon $\left(2a_1 \cos 2\pi \frac{\nu}{v} \cdot \frac{r_1 - r_2}{2}\right)^2$, so that the importance of the

phase difference, viz. $2\pi\nu\left(t - \frac{r_2}{v}\right) - 2\pi\nu\left(t - \frac{r_1}{v}\right)$ or $2\pi\frac{\nu}{v}(r_1 - r_2)$, as a determining factor in the estimation of the light intensity at any point will be realized.

If the waves arrive at the point in step, i.e. crest coinciding with crest, the principle of superposition states that the effect measured by y will be large, and the opposite occurs if the crest of one and the hollow of the other coincide. Between these two extreme cases we have a continuous range of possible values of y.

If the waves pass through regions of different refractive indices, the quantity which determines the magnitude of y is $\left(\Sigma\frac{r}{v}\right)_1 - \left(\Sigma\frac{r}{v}\right)_2$, i.e. the sum of r/v for one path minus the sum for the other. If we introduce the refractive index, we write $v = v_0/n$, and the determining quantity is Σnr since v_0 is common to both. Thus the above difference is the difference in optical paths. For a maximum effect we require that $\pi\frac{\nu}{v_0}(\underset{1}{\Sigma nr} - \underset{2}{\Sigma nr})$ should be equal to an integer times π (8.2) or that $\dfrac{\underset{1}{\Sigma nr} - \underset{2}{\Sigma nr}}{\lambda_0}$ should be an integer, where λ_0 is the wave-length in vacuo.

Difference of Optical Path for Rays refracted at a Spherical Surface

In fig. 8.3 AP represents a spherical surface of radius r, separating two media of indices n and n'.

From a point O in the first medium on the axis draw a line OP cutting the surface at P, and draw PQ in the second medium cutting the axis at Q. We shall not at first suppose that OP and PQ are conjugate rays, but shall calculate the difference between the optical path [OPQ] and [OAQ].

$$[OPQ] = nOP + n'PQ,$$
$$[OAQ] = nOA + n'AQ.$$

We shall write $p = [\mathrm{OPQ}] - [\mathrm{OAQ}]$.

$$\mathrm{OP}^2 = \mathrm{OA}^2 + \mathrm{PA}^2 + 2\mathrm{OA}\,.\,\mathrm{PA}\cos\alpha,$$

$$\cos\alpha = \mathrm{AM}/\mathrm{AP} = \frac{\mathrm{AP}}{2r}.$$

Thus

$$\mathrm{OP}^2 = \mathrm{OA}^2 + \mathrm{PA}^2\left(1 + \frac{\mathrm{OA}}{r}\right).$$

Similarly

$$\mathrm{PQ}^2 = \mathrm{QA}^2 + \mathrm{PA}^2\left(1 - \frac{\mathrm{QA}}{r}\right).$$

We shall limit ourselves to cases where we can write PA as equal to PM ($=y$) with sufficient accuracy and thus

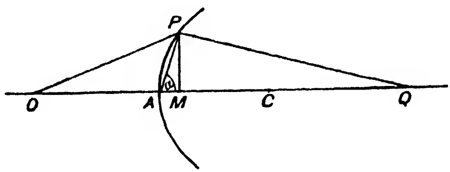

Fig. 8.3

can neglect terms containing powers of y greater than y^4. Writing $\mathrm{OA} = l$, $\mathrm{AQ} = \mathrm{L}$, we find

$$\mathrm{OP} = l\left\{1 + \frac{y^2}{2l}\left(\frac{1}{l} + \frac{1}{r}\right) - \frac{y^4}{8l^2}\left(\frac{1}{l} + \frac{1}{r}\right)^2\right\},$$

$$\mathrm{PQ} = \mathrm{L}\left\{1 + \frac{y^2}{2\mathrm{L}}\left(\frac{1}{\mathrm{L}} - \frac{1}{r}\right) - \frac{y^4}{8\mathrm{L}^2}\left(\frac{1}{\mathrm{L}} - \frac{1}{r}\right)^2\right\}.$$

Thus

$$p = \frac{y^2}{2}\left\{n'\left(\frac{1}{\mathrm{L}} - \frac{1}{r}\right) + n\left(\frac{1}{l} + \frac{1}{r}\right)\right\}$$
$$- \frac{y^4}{8}\left\{\frac{n'}{\mathrm{L}}\left(\frac{1}{\mathrm{L}} - \frac{1}{r}\right)^2 + \frac{n}{l}\left(\frac{1}{l} + \frac{1}{r}\right)^2\right\}. \quad . \quad 8.3$$

If y is so small that the term containing y^4 is negligible and if $\frac{n'}{L} + \frac{n}{l} = \frac{n' - n}{r}$, the value of p is zero.

Reference to equation (3.19) shows that this means that if we limit ourselves to the paraxial region the rays from the object arrive at the image all in the same phase.

If, however, rays fall on the refracting surface over a wider area and arrive at Q, they will have an optical path difference proportional to y^4, when Q is the paraxial image of O.

Spherical Aberration expressed in Terms of Difference of Optical Path

If a wave after refraction is spherical, it travels either towards a point O′ or away from it, O′ being its centre of curvature. Thus, if we refer to fig. 8.3, in the paraxial case a spherical wave from O after refraction will travel into the point Q, which in our notation is the point O′ in this case.

This condition is characteristic of image formation without aberration. If there are any causes of aberration such as the wave striking the outer non-paraxial zones, the emerging wave is no longer spherical. We must think of the paraxial zone as impressing upon the wave a definite curvature, turning the original wave from a sphere of one radius into a sphere of another radius, but the outer zones each alter the curvature in a way characteristic of the zone and so produce spherical aberration.

There will evidently be some relation between the difference in optical path and the intercepts by which we measured the spherical aberration in the last chapter. It is our object in this paragraph to find the relation.

Referring to fig. 8.4, let AB denote the sphere upon which the paraxial wave lies after refraction. This coincides with the actually emerging wave APS in the neighbourhood of A.

Let PS denote an element of this wave at a distance y

from the axis and let PQ be drawn perpendicular to this element. The element denotes the part of the wave which has traversed a zone of the refracting surface at distance y, and since it now travels in a uniform medium we can look upon PQ as the ray refracted by this zone. But the paraxial wave closes upon O′, and thus O′Q is the spherical aberration along the axis and O′T is the transverse aberration for this zone. Join PO′ cutting the

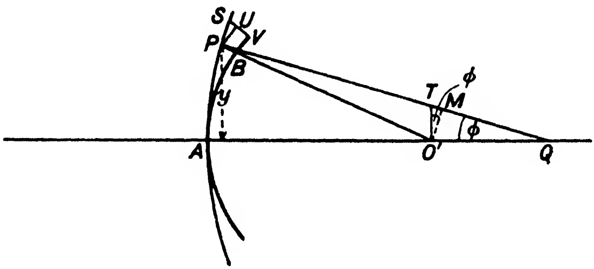

Fig. 8.4

sphere at B, and draw SV perpendicular to the sphere. Δp ($= n'$SU) is the difference of optical path for lines from P and S to O′. But if we denote PS by Δs, we have

$$\frac{\Delta p}{\Delta s} = n' \frac{\mathrm{SU}}{\mathrm{PS}} = n' \sin \mathrm{SPU}.$$

Since QP is perpendicular to PS and PU to PO′,

$$\sin \mathrm{SPU} = \sin \mathrm{QPO'} = \frac{\mathrm{O'M}}{\mathrm{O'P}} = \frac{\mathrm{O'T} \cos\phi}{\mathrm{O'P}} = \frac{\mathrm{O'Q} \sin\phi}{\mathrm{O'P}},$$

where $\angle \mathrm{O'QP} = \phi$ and O′M is perpendicular to PQ.

But if we limit ourselves to small deviations only and consequently to small values of y, we can write

$$\sin\phi = \frac{y}{\mathrm{PO'}} = \frac{y}{l'},$$

and thus from

$$\frac{\Delta p}{\Delta s} = n' \frac{\mathrm{O'Q}}{\mathrm{O'P}} \sin\phi = n' \frac{\mathrm{O'T}\cos\phi}{\mathrm{O'P}},$$

in the limit, when P and S are close together,

$$\mathrm{O'Q} = \frac{l'^2}{n'y}\frac{dp}{ds}.$$

Under these circumstances we can write $ds = dy$, the wave being confined to the neighbourhood of A, although out of the paraxial zone. Thus

$$\mathrm{O'Q} = \frac{l'^2}{n'y}\frac{dp}{dy}. \quad . \quad . \quad . \quad . \quad 8.4$$

For the transverse aberration in the case where we approximate by writing $\cos\phi = 1$, we have

$$\mathrm{O'T} = \frac{l'}{n'}\frac{dp}{dy}. \quad . \quad . \quad . \quad . \quad 8.5$$

It is interesting to obtain the formula (7.27) by an application of (8.3) and (8.4).

For two conjugate points O and O′,

$$p = -\tfrac{1}{8}y^4\left\{\frac{n'}{l'}\left(\frac{1}{l'} - \frac{1}{r}\right)^2 + \frac{n}{l}\left(\frac{1}{l} + \frac{1}{r}\right)^2\right\},$$

since now $\mathrm{L} = l'$.

Thus (8.4) gives

$$\mathrm{O'Q} = -\tfrac{1}{2}\frac{l'^2y^2}{n'}\left\{\frac{n'}{l'}\left(\frac{1}{l'} - \frac{1}{r}\right)^2 + \frac{n}{l}\left(\frac{1}{l} + \frac{1}{r}\right)^2\right\}.$$

But

$$\frac{1}{l'} - \frac{1}{r} = -\frac{n}{n'}\left(\frac{1}{l} + \frac{1}{r}\right).$$

Thus

$$\mathrm{O'Q} = -\frac{l'^2y^2}{2n'}\left(\frac{1}{l} + \frac{1}{r}\right)^2\left(\frac{n}{l} + \frac{n^2}{n'l'}\right),$$

and if we write, as before, $\mu = n'/n$, $\frac{1}{l} = \mathrm{L}$, $\frac{1}{r} = \mathrm{R}$, $\frac{1}{l'} = \mathrm{L}'$, we have

$$-\frac{\mathrm{O'Q}}{l'^2} = \frac{y^2}{2\mu^2}\left(\frac{1}{l} + \frac{1}{r}\right)^2\left(\frac{\mu}{l} + \frac{1}{l'}\right)$$
$$= \frac{y^2}{2\mu^2}(\mathrm{L} + \mathrm{R})^2(\mu\mathrm{L} + \mathrm{L}').$$

Previously we wrote $\Delta\mathrm{L}' = -\frac{\mathrm{O'Q}}{l'^2}$, so that this formula is identical with (7.27).

Seidel's Theory of the Aberrations

The theory of the formation of images by paraxial rays takes into account the first power of the angle of inclination of the rays to the axis. If we include higher powers of the angles, rays outside the paraxial zone are represented in the formulæ. In other words, we are taking into account the contribution of oblique rays to the formation of images.

L. von Seidel made contributions to geometrical optics as early as 1855 which take into consideration rays which make angles with the axis so great that it is necessary to take into consideration the third powers of these angles, while higher powers may be neglected. His work results in the development of conditions which must be satisfied in order that the aberrations we have discussed may be eliminated.

We shall give some account of the theory here, but by the method of the optical path discussed in this chapter, and we shall take the case of a refracting spherical surface to illustrate the theory.

In fig. 8.5 let AP represent the spherical surface separating two media and let SAC denote the axis of this optical system. A point source of light is situated at O at a distance h from this axis. If we draw the line OC, the paraxial image of O is situated at O′ on this line. Suppose that a small pencil of rays, with principal ray

OP, falls on the surface near P, and let the pencil be limited by a small stop S. For any ray of the pencil such as OP_1 the optical path difference $[OP_1O'] - [OA_1O']$ is given by formula (8.3) and can be written in the form

$$p_{P_1} = c_2 y^4 = c_2(P_1M_1)^4 \quad . \quad . \quad . \quad 8.6$$

to the order of accuracy considered in establishing this result. We must consider an expression for p in the case

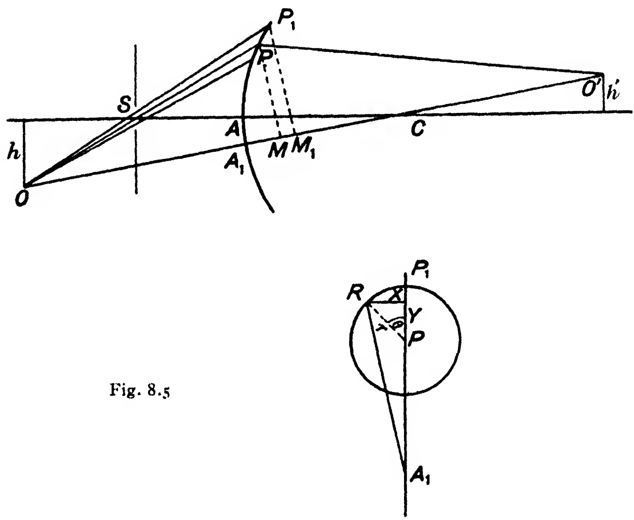

Fig. 8.5

of rays lying out of the plane of the figure. In order to obtain an approximate value of the perpendicular distance corresponding to P_1M_1, we shall suppose that the pencil of rays is in the form of a small cone of circular section and that it cuts the sphere at an area sufficiently near to A to allow us to consider the area to be a small flat circle. In fig. 8.5 this is shown with centre P and radius PP_1. The curved line P_1A_1 is drawn as a straight line in this figure, and it will be taken as equal to the perpendicular from P on to the line OO′, since this

perpendicular occurs in our formula in the fourth power; this is legitimate when the inclination of the rays to AC is not beyond the limits considered. Similarly, if R denote the intersection of a ray with the surface not in the plane of the figure, RA_1 may be taken as the perpendicular. In this case

$$p_R = c_2(A_1R)^4. \quad . \quad . \quad . \quad . \quad 8.7$$

Now

$$(A_1R)^2 = (A_1P + Y)^2 + X^2,$$

where the co-ordinates of R with respect to P are denoted by (X, Y). A_1P will be of the same order of magnitude as h, and we can write $A_1P = ah$, where a does not contain h.

Thus writing

$$p = p_R - p_P,$$

we obtain for the difference of optical path of the rays arriving at O′ from R and P the value

$$\begin{aligned} p &= c_2[\{(ah + Y)^2 + X^2\}^2 - a^4h^4] \\ &= c_2\{(X^2 + Y^2)^2 + 4aY(X^2 + Y^2)h \\ &\qquad + 2a^2(X^2 + 3Y^2)h^2 + 4a^3Yh^3\}. \end{aligned}$$

This result is expressed in terms of the displacement h of the object point, but for a fixed position of O it is clear that $\frac{h'}{h}$ is a constant ratio, and we can thus rewrite this equation in the form

$$\begin{aligned} p = a_1(X^2 + Y^2)^2 + a_2Y(X^2 + Y^2)h' \\ + a^3(X^2 + 3Y^2)h'^2 + 4a_5Yh'^3, \quad 8.8 \end{aligned}$$

h' being the distance from the axis of the point of intersection of the central emergent ray with the image plane.

If there is a succession of refracting surfaces, each will contribute a term of this form to p, where X and Y refer to the particular surface and h' to the image formed

immediately after refraction in it. Actually the final formula is similar to (8.8).

In order to interpret this result in terms of ray intercepts in the plane of the image, we make use of equation (8.5). In the case where we require the intercept parallel to the X direction, we use the formula

$$(\mathrm{O'T})_{\mathrm{X}} = \frac{l'}{n'}\frac{dp}{d\mathrm{X}},$$

Y remaining constant in this differentiation.

Similarly

$$(\mathrm{O'T})_{\mathrm{Y}} = \frac{l'}{n'}\frac{dp}{d\mathrm{Y}},$$

these displacements being measured in the appropriate directions from O′.

It should be noted that the optical path difference p is measured here as the difference between [ORO′] and [OPO′], so that the displacements are displacements from the point where the ray OP cuts OC after refraction.

$$(\mathrm{O'T})_{\mathrm{X}} = \frac{l'}{n'}\{4a_1\mathrm{X}(\mathrm{X}^2 + \mathrm{Y}^2) + 2a_2\mathrm{XY}h' + 2a_3\mathrm{X}h'^2\} \quad . \quad . \quad . \quad 8.9$$

and

$$(\mathrm{O'T})_{\mathrm{Y}} = \frac{l'}{n'}\{4a_1\mathrm{Y}(\mathrm{X}^2 + \mathrm{Y}^2) + a_2(\mathrm{X}^2 + 3\mathrm{Y}^2)h' + 6a_3\mathrm{Y}h'^2 + a_5h'^3\}. \quad . \quad 8.10$$

If we use polar co-ordinates, writing

$$\mathrm{X} = r\sin\theta,\ \mathrm{Y} = r\cos\theta,$$

we obtain

$$(\mathrm{O'T})_{\mathrm{X}} = \frac{l'}{n'}\{4a_1r^3\sin\theta + a_2r^2\sin 2\theta h' + 2a_3r\sin\theta h'^2\}, \quad . \quad . \quad . \quad 8.11$$

$$(\mathrm{O'T})_{\mathrm{Y}} = \frac{l'}{n'}\{4a_1r^3\cos\theta + a_2r^2(2 + \cos\theta)h' + 6a_3r\cos\theta h'^2 + a_5h'^3\} \quad . \quad 8.12$$

$$p = a_1r^4 + a_2r^3\cos\theta h' + a_3r^2(2 + \cos 2\theta)h'^2 + a_5r\cos\theta h'^3. \quad . \quad . \quad . \quad . \quad 8.13$$

Spherical Aberration

Suppose $h' = 0$ so that O′ lies on the optical axis, then the transverse aberrations are proportional to r^3, i.e. to the distance the ray cuts the surface from the axis. This result is to be compared with that expressed in equation (7.47).

Coma

In order to study the displacement proportional to h', let us represent the components graphically, writing

$$C_X = \frac{l'}{n'} a_2 r^2 \sin 2\theta h' \quad \text{(see 8.11)},$$

$$C_Y = \frac{l'}{n'} a_2 r^2 (2 + \cos 2\theta) h' \quad \text{(see 8.12)}.$$

For a particular value of r, i.e. for rays striking the refracting surface at points on a particular circle about P as centre, we can write

$$C_X = A \sin 2\theta, \;\; C_Y = 2A + A \cos 2\theta,$$

where

$$A = \frac{l'}{n'} a_2 r^2 h'.$$

The least value of C_Y is A and the greatest 3A, and the values for different angles θ (see fig. 8.7) are represented on a circle of radius A, as in fig. 8.6, where OA = A. This figure is described as the coma figure. Referring to fig. 8.4, we see that the point A corresponds to T of that figure when O′T = A, the displacement perpendicular to the plane of the figure, viz. C_X, being then zero. We can describe the point T in that case as T_A, and it is clear that $O'T_A = A$. It can be shown that the disturbances reaching this point from the ring about P (fig. 8.5) reinforce one another at this point, so that it may be regarded as a focus for this ring, the intensity being relatively great.

In order to show this, we require to find the optical

path difference for two points separated by a distance k parallel to the direction of Y in the image plane. Thus in fig. 8.7 we require the optical path difference [RO′ − RN]. In this figure it is supposed that the circle about P is drawn perpendicular to the plane of the paper. From R draw RK normal to the horizontal diameter of this circle. Then the points R, K, O′ and N lie in the same plane, and we draw NQ perpendicular to RO′ in this plane. Now

$$\mathrm{RO'} - \mathrm{RN} = k \sin \mathrm{QNO'} = k \cos \mathrm{QO'N}$$
$$= k \sin \mathrm{RO'K} = k \frac{\mathrm{RK}}{\mathrm{RO'}} = k \frac{r}{l'} \cos \theta.$$

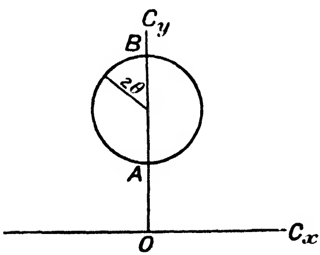

Fig. 8.6

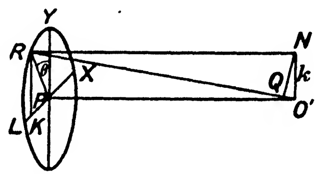

Fig. 8.7

If we refer to (8.13), we note that p is a measure of the path difference of the rays RO′ and PO′, O′ lying at a distance h' above the axis. We require to find the change in this quantity when O′ is replaced by a point N at distance k above O′.

If k be regarded as a quantity of the first order, the difference PN − PO′ is of the second order. Thus the change in value of (RO′ − PO′) consequent on displacing O′ to N is $-k \frac{r}{l'} \cos \theta$, since RO′ > RN. Thus the marginal path difference RN minus the central path distance PN now amounts to

$$\left(a_2 r^3 h' \cos \theta - \frac{n'}{l'} kr \cos \theta\right).$$

When the point N is T_A and consequently $k = \frac{l'}{n'} a_2 r^2 h'$, this difference vanishes. Thus from all points on the circle of radius r about P the wavelets arrive at T_A without phase difference. This is true for other values of r, T_A lying for these values at different distances from the axis, OT_A being proportional to r^2. The intensity falls off to both sides of these points, and we thus get a patch of light extending from O′ towards the farthest point A which corresponds to the maximum ring of the emergent pencil. The patch is pear-shaped and lies to one side of the image, and results from this aberration which is called coma.

Astigmatism

If we refer again to equations (8.11) and (8.12), the terms in h'^2 are

$$A_X = 2a_3 \frac{l'}{n'} r \sin\theta h'^2 = a \sin\theta \text{ (say)},$$

$$A_Y = 6a_3 \frac{l'}{n'} r \cos\theta h'^2 = 3a \cos\theta.$$

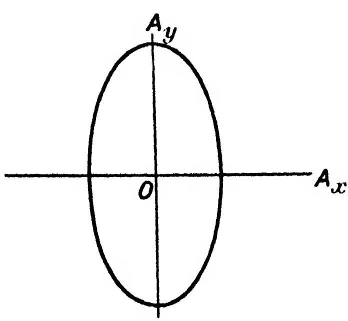

Fig. 8.8

On plotting an astigmatic figure (8.8) corresponding to the coma figure (8.6), we obtain the ellipse

$$\frac{A_X{}^2}{a^2} + \frac{A_Y{}^2}{9a^2} = 1,$$

the A_Y-axis being three times the A_X-axis.

Thus the rays will cut the image surface in an ellipse of this character.

Distortion

The displacement proportional to h'^3 is independent of the radius r, so that a stop, however small, does not avoid it, and it occurs in the Y-direction only. This displacement produces distortion, for it is proportional to the cube of the linear dimensions and is thus much increased at the limits of the image, so that the scale is not constant over the whole field.

The Five Aberrations

The discussion of this chapter is not quite complete, for the theory of von Seidel takes account of the five aberrations: spherical aberration, coma, astigmatism, curvature of field and distortion. Our theory has not included the fourth of these, but we have considered it in the last chapter. The complete theory of von Seidel contains an examination of the corrections that have to be made when the ray inclinations to the axis are so great that we have to take into account third powers as well as the first. The corrections are expressed in the form of five sums denoted by S_1, S_2, . . ., S_5, and the final result can be stated as follows. The image of a plane object will be true, i.e. sharply defined, flat and undistorted, if, and only if, all the sums are zero.

If $S_1 = 0$, the error known as spherical aberration is corrected, the vanishing of S_2 removes coma, and so on in the order of aberrations above enumerated.

The process is somewhat similar to removing the contributions to p in our formula (8.13).

CHAPTER IX

APERTURES AND PHOTOMETRY

It will have become evident in the previous chapters of this book that the limitation of the divergence of pencils of rays has important consequences in the character of the image. We have seen, for example, that in order that the image and object may be similar it is necessary to place two stops at points so that the condition (7.68) may be satisfied.

We can illustrate the effect of stops by taking two lenses (fig. 9.1) with a single stop S between them. Let S_1 be the image of S formed by the lens to the left and S_2 the image formed by the lens to the right. S_1 and S_2 are conjugate planes for the system of two lenses, and it is clear that only rays from a source O, which are directed through the image of the stop S, can pass through the system. The divergence of rays from O is thus limited to an angle $2U$ (see figure). The angle $2U$ is called the angular aperture of the system with respect to O, and $2U'$ is the angle of projection. In the eye we have an adjustable stop known as the iris, and between it and the outside is situated an optical system consisting of the cornea and the contents of the anterior chamber, i.e. the aqueous humour. On looking into an eye we see the pupil, which is the image of the iris formed by this interposed optical system. We take over this nomenclature and apply it to optical systems; in our example the hole in S is called the iris and that in S_1 is the pupil. On account of the reversibility of rays the space of S_2 is also a pupil, and we distinguish between the pupils by calling them entrance and exit pupils.

The rims surrounding the lenses are themselves stops, and in complex systems there may be a number of these limiting apertures. In order to find the entrance pupil in such cases, we proceed by constructing the image of every aperture made by that part of the system which

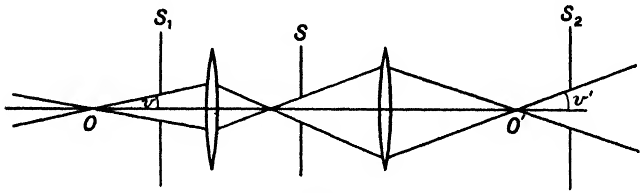

Fig. 9.1

lies in front of it. The smallest of these images is the entrance pupil. The exit pupil is the image of this aperture by the part of the system lying to the right, and these two pupils limit the rays through the system. These pupils are a conjugate pair for the whole system.

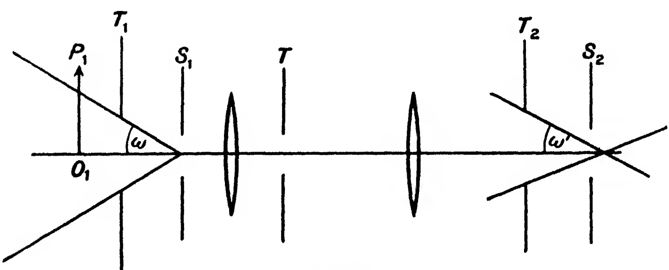

Fig. 9.2

Suppose that another stop T between the lenses has an image T_1 made by the lens to the left, and T_2 by that to the right. Let the aperture in T_1 subtend an angle 2ω at the centre of the entrance pupil. The field of view is limited to the cone of the smallest apex angle 2ω produced by a stop such as T. The apex angle of the emergent cone is limited to the value $2\omega'$ (fig. 9.2). It is

clear that the entrance pupil is not constant for all positions of the object, for generally, when there are several apertures, the greatest limitation to the divergence from another point O on the axis is not always produced by the same iris. The field of view stop will produce limitations of a rather complex character, as may be seen by considering pencils of rays from the object O_1P_1. From the central portion each point can send its full cone of rays through S_1, but this is not the case from the points lying towards the extreme part of the object. It is an advantage to arrange that T_1 lies in the plane of the object for which S_1 is entrance pupil, for then each point sends its full complement of rays through the instrument, and the final image is of uniform brightness to its outer borders.

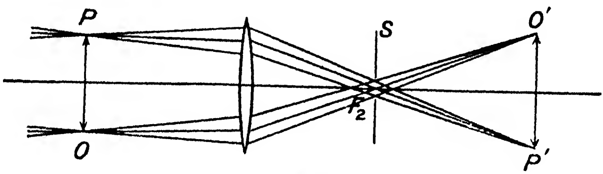

Fig. 9.3

Thus two of the effects of apertures are to limit the divergence of the beam and the field of view. The positions of the pupils determine the principal rays of cones leaving the object points, i.e. the rays proceeding to and from the centres of these pupils.

An interesting example of the use of apertures is in the production of infinitely distant pupils. Thus if we place the iris behind the first lens of a system so that it lies at the principal focus, the principal rays of the object space are all parallel to the axis.

Fig. 9.3 illustrates this case, and the system is described as telecentric on the side of the object. The use of this system in connection with micrometer microscopes is easily appreciated. If we require the measurement of the object OP, a small error in the focussing of the optical

system (e.g. the microscope) upon it is of no consequence since the size of the image O′P′ is determined by the extreme rays.

If the iris is placed at the first principal focus of the last lens of the system as in fig. 9.4, the system is telecentric towards the image. This is of use when it is required to measure the image of OP, for the principal rays are parallel to the axis, and even if there is not exact focussing, there is no error in the dimension O′P′. Instead of a sharp image we have a blurred one in the plane

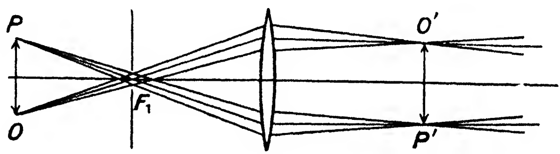

Fig. 9.4

of the cross hairs, but the size is correct. This system is of use in telescopes, the adjustment being made on the cross hairs with the aid of the eye-piece.

The Aperture and the Resolving Power

As we saw in the last chapter, when we investigate the intensity of light in the plane of the image, we proceed according to the principles of the wave theory. We apply the principle of superposition to all the wavelets arriving at any point, and the intensity is determined by the phase differences of the wavelets arriving.

Suppose that a spherical wave W falls on an aperture (fig. 9.5) with centre B′ and that it is converging to a point O′. We inquire what is the intensity at a neighbouring point P′, displaced a distance h' from O′ as in the figure.

We shall not give the calculation here, the subject belongs to the diffraction of light, but the result is of importance in the theory of optical instruments.

It is found that the intensity at the point P′ is proportional to the square of a quantity denoted by

$$\frac{\lambda d'}{\pi h' a} J_1\left(\frac{2\pi h' a}{\lambda d'}\right),$$

where the function J_1 denotes a Bessel function, λ is the wave-length of the light, a is the radius of the aperture, and d' is the distance B′O′.

This quantity vanishes for an infinite number of values of $\left(\frac{\pi h' a}{\lambda d'}\right)$. If we denote this quantity by x, the expression is $\frac{J_1(2x)}{x}$, and it vanishes for an infinite number of values of x. The smallest of these values is $x = 1{\cdot}92$.

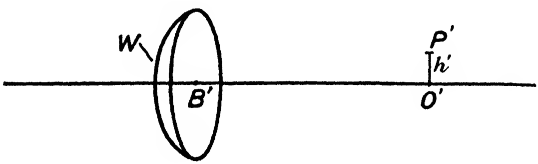

Fig. 9.5

This corresponds to the value of h' given by

$$h' = \frac{1{\cdot}92}{\pi}\,\frac{\lambda d'}{a} = 1{\cdot}22\,\frac{\lambda d'}{a}. \qquad 9.1$$

Thus extending from O′ outwards we find an illuminated area bounded by a dark ring of radius given by (9.1). As we proceed further the illumination increases and again falls to zero, according to the graph (1) of fig. 9.6. But the intensities fall off rapidly from these maxima.

We find an application of this result in the case of a telescope, for if the aperture is the entrance pupil of the instrument, O′ becomes a principal focus of the object glass, and the illumination in the focal plane is such that a bright patch of light is bounded by a dark ring of the radius given by (9.1), with f' written for d', where f' is the image focal length of the object glass.

The bright patch is spoken of as the Airy disc, in honour of Airy's work in this connection, the formula (9.1) being also known by his name.

Suppose that an object at O_1 (fig. 9.7) gives rise to an image at O_1' by refraction in the object glass of a tele-

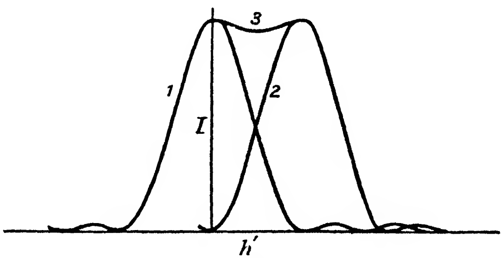

Fig. 9.6

scope. Then the illumination in the image plane through O_1' will be in accordance with our description. Suppose that a point O_2 gives rise to an image O_2', then we can imagine the superposition in the plane through $O_1'O_2'$ of two illuminations similar to that pictured in fig. 9.6, the curve for O_2 being displaced to the right.

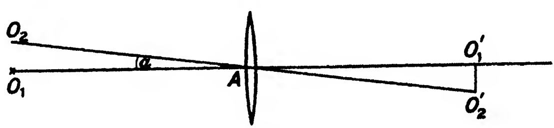

Fig. 9.7

By summation we can obtain the total effect of both sources. If they are too close together, it will be impossible to recognize the separate maxima, and it will thus be impossible to recognize that the light reaches the telescope from two sources. The instrument fails to "resolve" the two objects. If the maximum of the second curve falls over the first minimum of the first, the summation between the maxima is represented

by (3). It may be supposed that the depression in this curve will correspond to the power of the eye to appreciate the falling off of intensity which it denotes. But whether this be so or not, this amount of separation of the maxima has been adopted, following a suggestion of Lord Rayleigh's, as a measure of the resolving power of the object glass. When this separation is brought about, we have from (9.1)

$$O_1'O_2' = 1{\cdot}22\,\frac{\lambda f'}{a},$$

assuming that the objects are at a considerable distance from the lens. But $O_1'O_2'/f' = \alpha$ (fig. 9.7) when the angle is small, as is the case here. Thus

$$\alpha = 1{\cdot}22\,\frac{\lambda}{a}, \quad . \quad . \quad . \quad . \quad . \quad . \quad 9.2$$

and this formula then measures the smallest angle subtended at A by two objects, which can be resolved. It is clear that α is inversely proportional to the radius of the aperture, a fact which accounts for the large apertures of astronomical telescopes, which are required to separate objects subtending minute angles at the object glass.

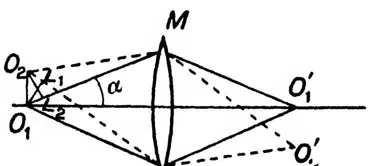

Fig. 9.8

This discussion applies especially to telescopes. The case of the microscope offers special features, for example the angle of the cone of rays falling on the object glass is no longer small (fig. 9.8).

Let us suppose that O_1 and O_2 are two points situated in front of the object glass of a microscope. It is important to remember that in practice the points are not in-

dependent sources of light like two stars seen through a telescope, to which our calculation of the resolving power has just been applied. The object in the case of a microscope is illuminated from outside and it is very important, as Abbe has pointed out, to take into consideration the way in which the object is illuminated. The determination of the separation O_1O_2, which is necessary in order that the appearance in the image plane shall be such that the points can be recognized as distinct, has been the subject of rather complicated analysis and of careful experiment, the most important contribution being due to Abbe. The result is that the finest detail that a microscope can resolve is determined by

$$O_1O_2 = \frac{\lambda}{2 \sin \alpha}.$$

The space between the object and the microscope objective is sometimes filled with a more highly refracting medium than air, such as oil. The same formula applies but the wave-length is that in the medium, so that, if we suppose that λ is the wave-length in air, the above limit of resolution can be written in the more general form:

$$O_1O_2 = \frac{\lambda}{2n \sin \alpha},$$

where n is the refractive index of the medium.

The quantity $n \sin \alpha$ is described as the numerical aperture, so that the result may be expressed as

$$\text{Limit of resolution} = \frac{\lambda/2}{\text{numerical aperture}}. \qquad 9{\cdot}3$$

Abbe examined the problem by studying an object in the form of a diffraction grating. If such an object be illuminated by parallel light, a series of spectra is observed in the focal plane of the objective, and it was shown that all the spectra must be gathered together and must take part in the formation of the retinal picture if a perfect

image is to be formed. By means of this criterion Abbe showed that the limit of resolution was one-half of that given in (9.3). But the use of a condenser to illuminate the object can be made to increase the value, almost doubling it, and thus (9.3) gives the limit of resolution of the microscope and condenser. The application of this line of argument to an object depends on the fact that the fine structure of the object gives rise to a diffraction pattern.

We can deduce the formula by a method due to Rayleigh depending upon the criterion he proposed, but the argument depends upon the assumption that the points are self-luminous, and we know that this is not the case. If the points O_1' and O_2' denote the central maxima of O_1 and O_2 respectively and if the minimum of O_2 lies at O_1', we know that the average difference of path of the rays from O_1 and O_2 to O_1' is $\lambda/2$. Thus the difference between O_2MO_1' and O_2NO_1' is λ and similarly for O_1MO_2' and O_1NO_2'. From the former we obtain

$$[O_2N] - [O_2M] = \lambda.$$

Drop perpendiculars from O_1 and O_2 on to O_2N and O_1M respectively.

$$O_2N = O_2L_1 + L_1N, \quad O_2M = O_1M - O_1L_2,$$
$$O_2N - O_2M = O_2L_1 + O_1L_2 + (L_1N - O_1M).$$

Now L_1N and O_1N may be taken as equal, since $\angle O_2NO_1$ is small and O_1 is assumed symmetrical with respect to the lens, so that $O_1M = O_1N$. Hence

$$O_2N - O_2M = O_2L_1 + O_1L_2 = 2O_1O_2 \sin\alpha.$$

Thus

$$2nO_1O_2 \sin\alpha = \lambda$$

or

$$O_1O_2 = \frac{\lambda}{2n \sin\alpha}. \qquad \dots \qquad 9.3$$

Photometry

Another effect of apertures is their influence on the brightness of the image formed by the optical system with which they are combined. In order to study this effect it is necessary to make certain definitions and obtain deductions from them, and we shall now consider these.

Suppose that an element of volume δv is surrounded by a unit sphere with its centre in the element. A certain amount of energy per second will be emitted from it when it is radiating, and the whole of it will pass through the spherical surface. We shall denote the amount by $A\delta v$, which is equivalent to a statement that the energy radiated by unit volume in unit time is equal to A. If we assume that the energy is emitted uniformly in all directions, the amount which passes down a cone of solid angle $\delta\omega$ is $\frac{A}{4\pi}\delta v\,\delta\omega$. If this uniformity does not exist, the expression must be written in the form $B_\omega \delta v\,\delta\omega$, where $\int B_\omega d\omega$ denotes the total emission of energy per second, i.e. $A = \int B_\omega d\omega$.

We shall not be concerned with the distribution of energy amongst the wave-lengths of the radiation nor between the states of polarization, so that it will not be necessary to consider the further division of B_ω, e.g. in the form $\int C_\lambda d\lambda$.

It is strictly incorrect to speak of emission of radiation from a surface, but in our applications of these ideas we are concerned with emission of radiation through a surface. It is thus convenient to speak of radiation from a surface when we mean radiation coming through it, the surface being either a real one or one imagined to exist in a space.

Thus in the formulæ the element of volume will be replaced by a surface element δS, and the radiation through it passing down a cone of solid angle $\delta\omega$ will be of amount $B_\omega \delta S\,\delta\omega$.

It is to be expected that B_ω will depend on the direction of the axis of the cone measured with respect to the normal to δS. This means that B_ω will depend upon θ (fig. 9.9). The determination of the form of this dependence is obtained from a simple observation.

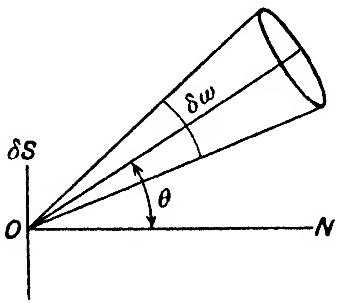

Fig. 9.9

A spherical surface emitting light appears equally bright in all parts when it is viewed from a point not too close to it. This can be interpreted if cones of equal solid angles at the point O contain the same quantities of energy wherever they cut the sphere.

Let OPQ (fig. 9.10) denote a cone of solid angle $\delta\omega$ cutting the surface in PQ. We suppose that O is sufficiently distant to make it possible to neglect differences

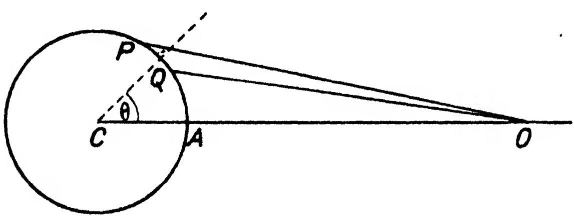

Fig. 9.10

such as OP − OA, i.e. O may be regarded as equally distant from all parts of the surface. The difference in the case now considered and that where the cone has its base near A is that the radiation from PQ is emitted to O in a direction making an angle θ with the normal to the surface. Thus the cone with angle $\delta\omega$ cutting the surface PQ receives light from a larger area of the sphere

than that of the normal cone with its base about A. The ratio of the two areas is as 1 : $\cos\theta$, but since the total quantity of light is the same in both cases, the amounts from unit area are in the ratio $\cos\theta$: 1. Thus if we change the notation and write I_0 for the amount of light per unit area emitted normally per unit solid angle, $I_0 \cos\theta$ denotes the amount emitted in the direction inclined at θ to the normal, and our expression for the energy emitted per second becomes $I_0 \cos\theta\, \delta S\, \delta\omega$. The quantity I_0 is described as the intensity of radiation of the surface.

Emissive Power

In fig. 9.11 let δS denote an element of a surface radiating from one side. In order to find the total radiation into a cone of apex angle 2U, surround the element with a unit sphere and let the arc PQ on the sphere be supposed to rotate about the normal to δS. It will cut out an area of magnitude $2\pi \sin\theta\, d\theta$ on the unit sphere, so that this expression measures the solid angle between two cones of semi-apex angles θ and $\theta + d\theta$. The amount of energy radiated within this solid angle is

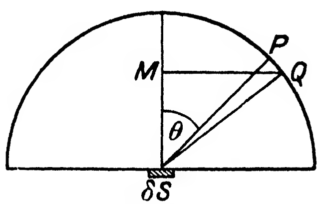

Fig. 9.11

$$I_0 \cos\theta\, \delta S 2\pi \sin\theta\, d\theta.$$

On integrating this between the limits 0 and U, we obtain

$$\pi I_0 \sin^2 U\, \delta S$$

for the energy radiated by δS in the finite cone.

If $U = \pi/2$, the expression is a measure of the energy radiated from δS, and its magnitude is

$$E\,\delta S = \pi I_0 \delta S.$$

The quantity $E = \pi I_0$ is called the emissive power of the surface and denotes the energy radiated per square centimetre per second.

Radiation from one Surface to another

In fig. 9.12 let δS be a radiating surface element of intensity of radiation I_0. If $\delta S'$ subtends a solid angle $\delta\omega'$ at δS, the amount of energy received by $\delta S'$ is

$$I_0 \delta S \cos i \, \delta\omega' = I_0 \delta S \cos i \, \frac{\delta S' \cos i'}{r^2}.$$

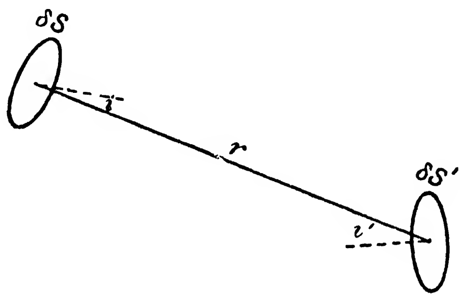

Fig. 9.12

From the symmetry of this expression it follows that the rôle of radiating surface may be interchanged, and $\delta S'$ receives from δS as much as δS would receive from $\delta S'$ if the latter radiated with intensity I_0. By integration the same is true of surfaces of finite extent.

Dependence of the Intensity on the Medium

Let us suppose that two media in the form of plates (1 and 2 in fig. 9.13) are of infinite extent and enclosed between two parallel plates also of infinite extent and perfectly reflecting on their outer surfaces. By this means we obtain an isolated system since no radiation can cross perfectly reflecting surfaces, and we obtain a simple geometrical configuration without end effects.

Let the inner surfaces of the plates be supposed to be coated with a substance which absorbs all the radiation incident upon it. Such a surface is described as being a perfectly black surface. We shall suppose that the intensity of radiation from the lower surface is I_0 and from the upper surface I_0'. The refractive index of the lower medium will be denoted by n and of the upper by n'.

When the system is in a state of equilibrium, each element of the bounding surfaces will be emitting and receiving energies in equal amounts per second. We shall consider the energy as passing to and from δS along rays, which are actually the principal rays of cones with their apices in δS.

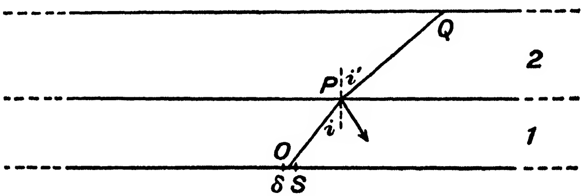

Fig. 9.13

Radiation in the spaces travels equally in all directions, so that every direction along which δS emits radiation is a direction along which it receives it. If we trace the energy flowing from δS along a cone in the direction making an angle i with the normal, i.e. falling on the surface separating 1 and 2 at an angle i, we note that it is partly reflected and partly transmitted at this boundary. Let the angle of refraction be i' and let r_i denote the reflecting power of the separating surface. Let r_i' denote the corresponding quantity applied to a ray falling from 2 upon 1. It can be shown that the reflecting powers r_i and r_i' are equal when the angles i and i' stand in the relation of angles of incidence and refraction.* As before,

* See, for example, *The Electron Theory of Matter*, O. W. Richardson, 2nd Edn., pp. 130, 132, or *Theoretical Physics*, W. Wilson, Vol. II, p. 172.

we can represent the element of radiation from δS at angles of incidence in the neighbourhood of i by

$$2\pi I_0 \sin i \cos i \delta i \delta S = \delta Q \text{ (say).}$$

Of this an amount $(1 - r_i)\delta Q$ passes into 2 and is ultimately absorbed by the bounding plate, and a fraction $r_i \delta Q$ is reflected and is absorbed by the lower plate. An equal amount $r_i \delta Q$ is restored to δS from elements of the lower plate. The loss to medium 2 is compensated by emission from the upper surface, the amount being a fraction $(1 - r_i')$ of

$$2\pi I_0 \sin i' \cos i' \delta i' \delta S = \delta Q' \text{ (say).}$$

We use the same symbol for the element δS in the case of the other surface since, on symmetrical grounds, there is an exact correspondence between the surface elements. Equilibrium requires

$$\int dQ = \int r_i dQ + \int (1 - r_i') dQ', \quad . \quad 9.4$$

where the integrals are made for all angles i and i', between which the relation $n \sin i = n' \sin i'$ exists. Thus the limits for i being 0 to $\pi/2$, the limits for i' are 0 to arc $\sin n/n' = c$ (say).

(9.4) can be written

$$I_0 \int_0^{\pi/2} (1 - r_i) \sin i \cos i \, di = I_0' \int_0^c (1 - r_i') \sin i' \cos i' \, di'. \quad . \quad . \quad . \quad 9.5$$

We shall neglect variation of n and n' with the wave-length in this calculation, although in the perfect radiator we have assumed all wave-lengths are necessarily present.

Transforming the right-hand side of (9.5) we obtain, since

$$\sin i' \cos i' \, di' = (n/n')^2 \sin i \cos i \, di,$$

$$\left(\frac{I_0}{n^2} - \frac{I_0'}{n'^2}\right) \int_0^{\pi/2} (1 - r_i) \sin i \cos i \, di = 0. \quad 9.6$$

The integral in this equation is not zero, hence

$$\frac{I_0}{n^2} = \frac{I_0'}{n'^2}. \qquad \ldots\ldots \quad 9{\cdot}7$$

This proof relates to the total radiation only and is limited by our assumption with regard to the refractive index. The equation does, however, hold for partial groups of waves of lengths extending from λ to $\lambda + d\lambda$, and we can write

$$\frac{I_{0\lambda}}{n^2} = \frac{I_0{}'_\lambda}{n'^2}, \qquad \ldots\ldots \quad 9.8$$

where n and n' refer to the wave-length λ and $I_{0\lambda}$ and $I_{01\lambda}$ are the corresponding intensities.*

Illumination of Images in Optical Systems

In fig. 9.14 let A and A′ denote the entrance and exit pupils of an optical instrument, which produces an image δS′ of δS. We shall suppose that all the light originating in δS is brought to a focus on δS′, i.e. all elementary cones

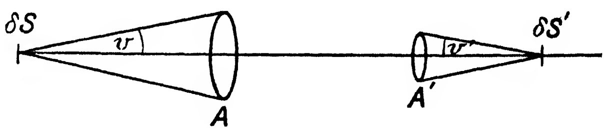

Fig. 9.14

from the object are transformed to cones with apices on the image. We can compare this with the conditions laid down in connection with the development of the sine condition in Chapter VII.

If the intensity of light from δS be I_0, the energy which enters the system from the object is $\pi I_0 \sin^2 U\, \delta S$, where 2U is the angle subtended at δS by a diameter of the aperture. If we neglect losses in the instrument, all this energy will reach the image. The image now becomes a source of radiation, and will radiate into a cone

* See *Theory of Optics*, Drude, p. 502.

of apex angle $2U'$, where $2U'$ is the angle subtended at $\delta S'$ by a diameter of the exit pupil. If the intensity of the image be I_0', the total amount of energy radiated is $\pi I_0' \sin^2 U' \delta S'$. Thus

$$I_0 \sin^2 U \delta S = I_0' \sin^2 U' \delta S'. \quad . \quad . \quad 9.9$$

If this be combined with (9.7), we obtain the sine-condition

$$n^2 \sin^2 U \delta S = n'^2 \sin^2 U' \delta S'. \quad . \quad . \quad 9.10$$

From (9.7) it follows that when the object and image lie in media of the same refractive index, the values of I_0 and I_0' are equal provided we neglect any losses by absorption. Since some loss is unavoidable in practice, it follows that I_0' is always less than I_0, or in any optical system the intensity of radiation of the image is never greater than that of the object when both lie in the same medium. We have already mentioned in connection with the resolving power of the microscope that an increase can be obtained in oil immersion systems where illumination is provided by a condenser which directs light on to an object placed in a highly refracting medium. In this case the intensity of radiation from the source is increased when it passes into the oil to illuminate the object. If the source is in air and the oil has an index n', this increase is according to the equation

$$I_0' = n'^2 I_0,$$

and the quantity of light entering the microscope is proportional to $n'^2 \sin^2 U$ (9.10), where $2U$ is the angle between the extreme rays entering the entrance pupil, i.e. proportional to the square of the numerical aperture. We must be careful to differentiate between the intensity of radiation and the intensity of illumination of light on a screen placed to receive the image.

It is clear that if we place a screen to receive light from the sun, for example, the amount of energy per unit area, i.e. the intensity of illumination, is less than the corre-

sponding quantity when an image is thrown on the screen by a lens or any optical system. In the case of an image of area $\delta S'$ the intensity of illumination is

$$\pi I_0' \sin^2 U' = \pi I_0 \sin^2 U',$$

if $n = n'$. The energy received by $\delta S'$ reaches it through the exit pupil, and this surface and $\delta S'$ stand in relation to one another in the same way as any two surfaces with interchange of radiation. In considering two radiating surfaces we saw that the rôle of radiator could be interchanged (fig. 9.12). If $\delta S'$ were radiating to the exit pupil, it would do so with intensity I_0 in the present case, thus the exit pupil may be regarded as radiating to $\delta S'$ with the intensity I_0 of the source of radiation. The effect of the system is thus the same as if $\delta S'$ were brought so near to the source that the latter subtended the angle $2U'$ at the element.

The Brightness of Images

When an observer looks at an object, the sensation of brightness is determined by the amount of energy falling on his retina per unit area. We shall suppose that there is no optical instrument between the object and the eye and that the object is a bright surface δS. Let this produce an image on the retina of area δS_0, and suppose that the object is situated in air and that the refracting medium of the eye has a refractive index n'. The pupil of the eye will be supposed to limit a cone of apex angle $2W_0'$, the apex lying on the retina. Thus since $I_0' = n'^2 I_0$, the amount of energy reaching the retina is

$$\pi n'^2 I_0 \sin^2 W_0' \delta S_0,$$

where I_0 denotes the intensity of radiation of δS. The amount per unit area is thus

$$H_0 = \pi n'^2 I_0 \sin^2 W_0'. \quad . \quad . \quad . \quad 9.11$$

If an object is at some distance from the eye, the size of the pupil changes very little with the distance, and (9.11) is a quantity almost independent of the position of the object. For this reason H_0 is a suitable quantity to adopt as a measure of the brightness of the source. It is described as the natural subjective brightness of the radiating surface.

The radiating surface may be the original source of the rays or it may be an image produced by an optical system. The word " natural " indicates that no instrument is situated between it and the eye.

The value of H_0 diminishes if the pupil decreases in size, i.e. if W_0' diminishes.

If an instrument is being used with the eye, reference to fig. 9.2 will show that if the pupil of the eye and the exit pupil S_2 are coincident, the full field of view is used. In the other case it is possible that the eye pupil may limit this field.

Suppose that a surface of area δS and intensity I_0 is situated at O (fig. 9.1) and that the instrument represented in this figure produces an image at O′ of area $\delta S'$, the medium on both sides being air. To an eye with its pupil at S_2 it is as if a source of intensity I_0 and area $\delta S'$ were situated at O′.

We suppose that the pupil of the eye is of a definite size and that W_0' is the semi-angle of the cone of rays in the eye.

If the exit pupil S_2 completely covers the pupil of the eye, the amount of light entering is $\pi n'^2 I_0 \sin^2 W_0' \delta S_0$. The amount of light from a source reaching the retina per unit area is described as the subjective brightness of the source. We have seen that if there is no optical instrument between the source and the eye, the brightness is described as " natural ".

Thus the subjective brightness of the object is $\pi n'^2 I_0 \sin^2 W_0'$, disregarding, of course, losses in the instrument. This quantity measures the natural brightness of the image formed at O′.

If, however, the exit pupil is smaller than the pupil

of the eye, the latter is only partly filled, let us say, by a cone with apex angle $2W'$ at the retina. The brightness is now

$$H = \pi n'^2 I_0 \sin^2 W'. \quad . \quad . \quad . \quad 9.12$$

The ratio

$$\frac{H}{H_0} = \frac{\sin^2 W'}{\sin^2 W_0'}. \quad . \quad . \quad . \quad . \quad . \quad 9.13$$

The angles are usually small, about $5°$, so that this ratio becomes

$$\frac{H}{H_0} = \frac{\text{area of exit pupil}}{\text{area of pupil of the eye}}, \quad . \quad 9.14$$

it being remembered that both pupils are in the same plane.

We can obtain an alternative form of (9.14) in terms of the magnification of the optical instrument, which is of importance in connection with the microscope. The alternative form, like (9.14), applies only when $H < H_0$.

Let the angle of projection U' of the instrument (fig. 9.1) be too small to cause the cone to fill the pupil of the eye, and let the image $\delta S'$ at O' be situated at a distance d from the exit pupil and hence at this distance also from the eye pupil. Then the area of the exit pupil is $\pi(d \tan U')^2$, or if U' is small this area may also be put equal to $\pi(d \sin U')^2$.

If the pupil has a radius p, we find from (9.14),

$$\frac{H}{H_0} = \sin^2 U' \frac{d^2}{p^2}.$$

But by the law of sines (9.10), if the object lies in a medium of refractive index n and the image in a medium of unit index,

$$n^2 \sin^2 U \, \delta S = \sin^2 U' \, \delta S'.$$

If the linear magnification of the system be m,

$$m^2 = \frac{\delta S'}{\delta S},$$

hence

$$\frac{\mathrm{H}}{\mathrm{H}_0} = \frac{n^2 \sin^2 \mathrm{U}}{m^2} \frac{d^2}{p^2}.$$

The numerical aperture of the system will be denoted by $a = n \sin \mathrm{U}$, thus

$$\frac{\mathrm{H}}{\mathrm{H}_0} = \frac{a^2}{m^2} \frac{d^2}{p^2}. \quad \ldots \ldots \quad 9.15$$

The Brightness of Star Sources

Distant stars subtend such small angles at the eye and at optical instruments that the foregoing conclusions no longer apply. Diffraction effects of the pupil have to be considered, and it follows from the application of the principles of physical optics that the size of an image depends upon the size of the pupil and not upon the magnification of the system.

If the light from a star falls on the eye, the amount of energy incident upon the retina is proportional to the area of the pupil and inversely proportional to the square of the distance of the source. We can define the natural brightness as before, and we have now $\mathrm{H}_0 \propto \frac{p^2}{r^2}$, where r is the distance of the source and p the radius of the pupil.

If the light falls on an optical system such as a telescope, the light entering is proportional to the square of the radius of the objective, and if all this light gets into the eye, i.e. if the exit pupil is not greater than the pupil of the eye, $\mathrm{H} \propto \frac{\mathrm{P}^2}{r^2}$, where P is the radius of the objective, i.e.

$$\frac{\mathrm{H}}{\mathrm{H}_0} = \frac{\mathrm{P}^2}{p^2}. \quad \ldots \ldots \quad 9.16$$

In this case the brightness for the instrument exceeds the subjective brightness in this ratio.

If the exit pupil exceeds that of the eye, the latter acts as the exit pupil and only part of the object glass is effective. By (6.51) the linear magnification is denoted by b, and since the entrance and exit pupils, in this case object glass and pupil of the eye, are conjugate, we have for the radius E of the effective part of the object glass

$$\frac{p}{E} = b.$$

Thus

$$\frac{H}{H_0} = \frac{E^2}{p^2} = \frac{1}{b^2}. \quad . \quad . \quad . \quad . \quad 9.17$$

Now by (6.55) the reciprocal of the linear magnification is equal to the angular magnification; thus the brightness in this case is equal to the square of the angular magnification multiplied by the natural brightness.

Now the background of the star is a luminous surface to which the preceding considerations to extended objects apply, and in consequence it follows that its brightness is never increased by the instrument. This contrast with the case for small objects makes it possible, especially with a large instrument in which P is great, to observe stars standing out more brightly against the background when viewed through a telescope than when viewed by the eye alone.

CHAPTER X

CHROMATIC ABERRATION

Images formed by optical systems are liable to a further defect which is incurred on account of the fact that the refractive index of a medium depends on the wave-length of light. Many sources of light give out rays of " white " light, which undergo dispersion when refracted. This is best studied by means of the deviation which a bundle of rays undergoes in passing through a prism. It is observed that the apparently uniform bundle of rays of white light emerge from the prism as bundles of coloured rays, the white light behaving as though it were analysed into coloured components.

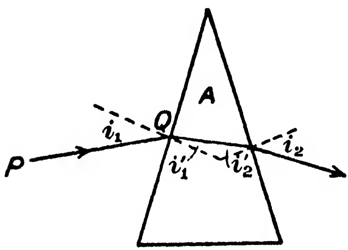

Fig. 10.1

We saw in equation (3.10) that the deviation of a ray through a prism is given by

$$D = i_1 + i_2 - A. \quad . \quad . \quad . \quad . \qquad 10.1$$

If the prism be thin and the angles concerned are all small (fig. 10.1), we can write

$$i_1 = ni_1', \quad i_2 = ni_2',$$

and since

$$i_1' + i_2' = A,$$

$$D = (n - 1)A. \quad . \quad . \quad . \quad . \qquad 10.2$$

If the original ray PQ contain two components for which the difference in refractive index is δn, the difference in deviation δD is given by

$$\delta D = A\delta n. \qquad \qquad 10.3$$

Thus

$$\frac{\delta D}{D} = \frac{\delta n}{n - 1}. \qquad \qquad 10.4$$

This expression is independent of the angle of the prism, and the quantity on the right refers to the medium alone. Thus it may be conveniently chosen as a measure of the power of the medium to separate the two rays of which the indices are $(n + \delta n)$ and n. We denote the dispersive power by ω, where

$$\omega = \frac{\delta n}{n - 1}. \qquad \qquad 10.5$$

In practice we generally refer this to two rays, red and blue, with refractive indices n_r and n_b, and in this case

$$\omega = \frac{n_b - n_r}{n - 1}. \qquad \qquad 10.6$$

Here n is the refractive index of the yellow rays of the spectrum, with a value equal approximately to the arithmetic mean of n_b and n_r.

Achromatic Combinations of Prisms

It is important that optical instruments should be corrected for this colour defect, in other words, they should show no chromatic aberration.

To illustrate the principle on which these corrections are based, consider two thin prisms (fig. 10.2) of different refracting materials and with different dispersive powers ω and ω'.

The effect aimed at is to produce equal deviations in the extreme rays, the red and the blue.

Thus a ray of white light incident on the system will give rise to coloured rays, the red and blue being parallel. They are not required to be parallel to the original ray, since it is the purpose of most optical instruments to produce deviation. If n and n' denote the refractive indices for yellow light in the two prisms with refracting angles A and A′, the mean deviation is

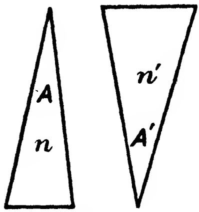

Fig. 10.2

$$(n - 1)A - (n' - 1)A'. \quad 10.7$$

The angle between the red and blue rays after refraction in the first prism is by (10.3) $A\delta n = A(n_b - n_r)$.

Similarly, the second prism produces a separation $A'(n_b' - n_r')$ in the opposite direction, and the total separation is

$$A(n_b - n_r) - A'(n_b' - n_r'). \quad . \quad . \quad 10.8$$

In order to make this zero it is necessary to choose the angles in the ratio

$$\frac{A}{A'} = \frac{n_b' - n_r'}{n_b - n_r}.$$

It should be noted that it does not follow that any other pair of dispersed rays emerge parallel to one another, but in practice the correction so obtained is sufficient for many cases. It is possible, by using more prisms, to bring about equal deviations for more than two colours.

Dispersion without Deviation

If the mean deviation (10.7) be made to vanish by the proper choice of A and A′, we can still have the separation between the red and blue rays given by (10.8). This principle is employed in the direct vision spectroscope, and results in the production of a spectrum without producing any average deviation in the beam.

Achromatism of Lenses

The possibility of combining two thin prisms to produce equal deviations in two rays of differing wave-lengths suggests that a pair of lenses, properly chosen, may have a similar property. If n denote the refractive index for a particular wave-length, the focal length of a thin lens is given by (4.5)

$$\frac{1}{f} = (n - 1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

Thus if n changes by δn, we find that the focal length changes by an amount given by

$$\delta\left(\frac{1}{f}\right) = \delta n\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \frac{\delta n}{n - 1} \cdot \frac{1}{f} = \frac{\omega}{f}. \qquad 10.9$$

The fact that the focal length is different for different wave-lengths means that an object emitting white light will give rise to a series of coloured images at different distances from the lens and of different sizes.

It is not always possible to remove the defect of different position and different size at the same time, and a choice has to be made in particular cases.

The Case of Two Thin Lenses in Contact

We shall consider the possibility of combining two lenses so that the combination has the same focal length for red as for blue rays.

If f_1 and f_2 denote the focal lengths for the mean rays of two lenses placed together, the focal length of the combination is given by (6.47) with $d = 0$,

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}. \quad . \quad . \quad . \quad . \qquad 10.10$$

We require that there should be no change in F for a change of δn in the refractive index, where $\delta n = n_b - n_r$.

From (10.10)

$$\delta\left(\frac{1}{F}\right)=\delta\left(\frac{1}{f_1}\right)+\delta\left(\frac{1}{f_1}\right)$$

$$=\frac{\omega_1}{f_1}+\frac{\omega_2}{f_2}.$$

Thus the condition is

$$\frac{\omega_1}{f_1}+\frac{\omega_2}{f_2}=0. \quad . \quad . \quad . \quad . \quad . \qquad 10.11$$

ω_1 and ω_2 are the dispersive powers of the media of two lenses respectively. The condition does not depend on the distance of the object or on its size, and the combination is achromatic, in the restricted sense of uniting the two colours, for all distances and sizes of the object.

The quantities ω_1 and ω_2 are essentially positive, thus f_1 and f_2 are of opposite signs. The lenses are thus convex and concave, and the combination in the telescope objective usually consists of a double convex lens of crown glass and a double concave lens of flint glass cemented together. In the objective of a microscope a plano-concave lens of flint glass is cemented to a double convex lens of crown glass, with the plane face toward the object.

The recombination of two colours in this way is sufficient for many purposes, but the achromatism may be improved by increasing the number of lenses, when it is possible to recombine as many colours as there are lenses.

Achromatism of Complex Systems

In the case of complex optical systems, if the system is to be achromatic it is necessary that the positions of the unit planes and the focal lengths shall be the same for the different colours. Since these positions and lengths depend on the refractive index, this is impossible.

We shall thus examine the special case of two separated thin lenses and study the extent to which it is possible to secure achromatism with them.

In fig. 10.3 let AB denote an object and A′B′ the image in the first lens L_1. Let A″B″ denote the final image. Then

$$\frac{AB}{A''B''} = \frac{BL_1}{B'L_1} \cdot \frac{B'L_2}{B''L_2}.$$

If the final coloured images corresponding to two rays are to be formed in the same position, $B''L_2$ must be the same for both images. If the coloured images are to be of the same size, A″B″ must be the same for both. From the equation last written down it thus follows that $\frac{B'L_2}{B'L_1}$ is

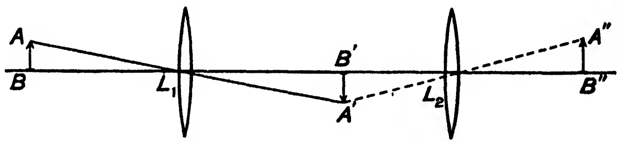

Fig. 10.3

the same for the two images or that A′B′ must be the same for them. This means that each lens must be achromatic for the two rays, and consequently each must consist of an achromatic pair in contact.

The focal length of two separated lenses is given by (6.44)

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{a}{f_1 f_2},$$

where a is the distance between the lenses.

If the colour of the light changes, f undergoes a change given by

$$\delta\left(\frac{1}{f}\right) = \frac{\omega_1}{f_1} + \frac{\omega_2}{f_2} - \frac{a(\omega_1 + \omega_2)}{f_1 f_2}, \qquad . \quad 10.12$$

and thus if f is to remain unchanged for the change of colour, the equation

$$\frac{\omega_1}{f_1} + \frac{\omega_2}{f_2} - \frac{a(\omega_1 + \omega_2)}{f_1 f_2} = 0 \qquad . \quad 10.13$$

must be satisfied.

When both lenses are made of the same refracting material, $\omega_1 = \omega_2$ and

$$\frac{1}{f_1} + \frac{1}{f_2} - \frac{2a}{f_1 f_2} = 0$$

or

$$a = \tfrac{1}{2}(f_1 + f_2). \qquad . \quad . \quad . \quad 10.14$$

When this relation is satisfied, the achromatism is true for all colours. Thus a pair of thin lenses separated by the distance given by (10.14) has the same focal length for all colours. The system is said to be achromatic with respect to the focal length.

If reference be made to (6.41) and (6.42) it will be noted that the principal points have different positions for different colours, since f_1 and f_2 vary when the refractive index changes.

The result is that both the position and the size of the image vary with the colour unless, as we have seen, both lenses are achromatic. This may be examined in detail by referring to fig. 10.3.

Let the object and image co-ordinates for the first lens be l_1 and l_1', and for the second l_2 and l_2'.

Then

$$\frac{AB}{A''B''} = \frac{l_1}{l_1'} \cdot \frac{l_2}{l_2'} = \frac{l_1}{l_1'}\left(\frac{l_2}{f_2} - 1\right)$$

$$= \frac{l_1}{l_1'}\left(\frac{a - l_1'}{f_2} - 1\right).$$

We can further express this in terms of l_1 only, and we obtain finally

$$\frac{AB}{A''B''} = 1 - \frac{a}{f_2} - l_1\left(\frac{1}{f_1} + \frac{1}{f_2} - \frac{a}{f_1 f_2}\right).$$

If we require that the image $A''B''$ shall remain unaltered

in size as the colour changes, say from blue to red, then we must satisfy the condition

$$l_1\left\{\frac{\omega_1}{f_1}+\frac{\omega_2}{f_2}-\frac{a}{f_1f_2}(\omega_1+\omega_2)\right\}+a\frac{\omega_2}{f_2}=0. \quad 10.15$$

The two images are not in the same position, and since l_1 occurs in the formula, it follows that when satisfied the result is true for one position only.

In spite of this limitation the result is useful since the eye is a better judge of size than of distance, and a better image is obtained if the sizes of the coloured images are equal. If the object is very distant l_1 is large, and the condition reduces to

$$\frac{\omega_1}{f_1}+\frac{\omega_2}{f_2}-\frac{a}{f_1f_2}(\omega_1+\omega_2)=0. \quad . \quad 10.16$$

If the lenses are of the same material we again obtain condition (10.14), and we thus conclude that when this condition is satisfied, the coloured images of a distant object are all of the same size.

These considerations apply to paraxial rays, and we have the added difficulty of spherical aberration when the inclination of rays to the axis becomes greater. Reference to the formulæ for spherical aberration shows that this effect depends on the refractive index. To obtain a good image with pencils of rays of large inclination, we have to correct for spherical aberration at least for two colours. If we wish to obtain an image of a surface element, the sine law (7.34) must also be satisfied for two colours. Such systems have been called " apochromatic " by Abbe, and we shall see that they are of importance in the construction of microscopes.

CHAPTER XI

OPTICAL INSTRUMENTS

We come now to the practical aim of our study, which is the construction of optical instruments for the purpose of producing images geometrically similar to the objects. If we put this in the language of geometrical optics, we may say that we aim at the construction of an optical instrument for which the image space is in collinear relation to the object space.

We have seen that there is no instrument for which this condition is satisfied completely, except the plane mirror, which we shall not regard as an optical instrument. It is true that the collinear relation exists for small inclinations of the rays to the axis, but this is of no practical importance, for the amount of energy entering a system by means of light rays satisfying this condition would produce but feeble illumination. In practice we must use rays of large inclination, and at once we see the limitation imposed, for we have shown that even a small object cannot give rise to an exactly similar image for all positions when the rays are no longer in the paraxial region. The foregoing chapters have shown the nature of the deviations from the collinear relation and have suggested how the defects may be to some extent overcome.

In actual instruments the refraction of the rays is distributed over various surfaces, and the effects of each and of their distances apart are considered in order that the principles we have been studying may be applied to arrive at the best possible instrument, i.e. one which

gives an image as nearly similar to the object as possible. Each instrument is considered in accordance with the nature of the work that is required of it.

The two optical instruments which come at once to mind are the telescope and the microscope. These are used for viewing distant and near objects respectively. Each of these instruments is a combination of simple instruments, and we may regard them for the moment as composed of the objective and the eye-piece. We require of the objective of a microscope that it shall produce an image in a small field of vision by wide-angled pencils, and this is true, though to a less extent, in the case of a highly magnifying telescope.

But of the eye-piece and similarly of a magnifying glass we wish to make the field of vision large, and to attain our object we sacrifice the width of the pencils.

The camera lens, however, is required to have a large field of view, and to form its images by wide-angled pencils. The latter is important, since we require a bright image to make rapid action possible.

In this chapter we shall consider these instruments, and in addition we must briefly study the eye, which in many cases acts as part of the optical system employed; e.g. in the case of the microscope we have as our complex system the objective, the eye-piece, and the eye. Moreover, we make use of comparisons of images seen by the eye unaided, and by the eye in combination with the optical instrument.

We begin with some considerations of a general character in the theory of optical instruments.

Formulæ referred to the Entrance and Exit Pupils

It is often convenient to refer measurements in the case of optical instruments to the entrance and exit pupils as origins. The general form of the equations is the same as that occurring in Chapter VI, since the pupils are conjugate planes (cf. 6.7).

The figure 11.1 is drawn to represent the standard case, and if EG is the entrance pupil, the position of its conjugate plane (E′G′) is obtained by the usual construction. We shall measure the object and image distances from E and E′, denoting them by p and p' respectively. If we denote the distance EF by g and E′F′ by g', then

$$\frac{g'}{p'} + \frac{g}{p} = 1. \qquad \ldots \qquad 11.1$$

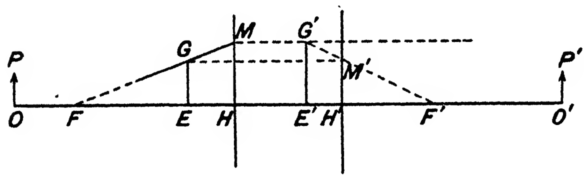

Fig. 11.1

Let the radii of the apertures be denoted by a and a'. Then

$$\frac{\mathrm{GE}}{\mathrm{EF}} = \frac{\mathrm{MH}}{\mathrm{HF}} = \frac{\mathrm{G'E'}}{\mathrm{HF}},$$

$$\frac{\mathrm{G'E'}}{\mathrm{E'F'}} = \frac{\mathrm{M'H'}}{\mathrm{H'F'}} = \frac{\mathrm{GE}}{\mathrm{H'F'}},$$

or

$$\frac{a}{g} = \frac{a'}{f}, \quad \frac{a'}{g'} = \frac{a}{f'}.$$

Thus

$$g = \frac{a}{a'}f, \quad g' = \frac{a'}{a}f'.$$

Thus 11.1 becomes

$$\frac{a'}{a}\frac{f'}{p'} + \frac{a}{a'}\frac{f}{p} = 1. \qquad \ldots \qquad 11.2$$

By (6.19)

$$\frac{y'}{y} = \frac{f' - l'}{f'} = \frac{f}{f - l}.$$

But

$$f' - l' = \text{H}'\text{F}' - \text{O}'\text{H}' = \text{E}'\text{F}' - \text{O}'\text{E}' = g' - p',$$

and similarly

$$f - l = g - p.$$

Thus

$$\frac{y'}{y} = \frac{g' - p'}{f'} = \frac{f}{g - p}$$

$$= \frac{\frac{a'}{a}f' - p'}{f'} = \frac{f}{\frac{a}{a'}f - p} \quad . \quad . \quad 11.3$$

$$= -\frac{a}{a'} \cdot \frac{f}{f'} \cdot \frac{p'}{p}. \quad . \quad . \quad . \quad . \quad . \quad 11.4$$

This follows from (11.3) by the application of (11.2).

Further, by means of (6.23),

$$\frac{y'}{y} = -\frac{a}{a'}\frac{n}{n'} \cdot \frac{p'}{p}. \quad . \quad . \quad . \quad . \quad 11.5$$

The form of equations (11.2) and (11.5) can be changed into one which is more convenient in some applications of them. Let us write

$$\text{Q} = \frac{na^2}{p}, \quad \text{Q}' = \frac{n'a'^2}{p'}, \quad \text{D} = \frac{n}{f} = \frac{n'}{f'} \text{ (cf. 6.23);}$$

(11.2) then becomes

$$\text{Q}' + \text{Q} = a'a\text{D} \quad . \quad . \quad . \quad 11.6$$

and (11.5) becomes

$$\frac{y'}{y} = -\frac{\text{Q}}{\text{Q}'} \cdot \frac{a'}{a}. \quad . \quad . \quad . \quad . \quad . \quad 11.7$$

Depth of Focus

Let E and E′ (fig. 11.2) denote the positions of the entrance and exit pupils of an optical instrument set so that an image of a plane object situated at O is brought to a focus on a screen at O′. Let O_1 and O_1' be another pair of conjugate points, and let $G'O_1'$ be an extreme ray cutting the screen at O′ in Z. Thus the extreme rays from an object at O_1, instead of giving rise to a point image O_1', as they would if the screen were placed there, will give rise to a " blur circle " on the screen, of radius O′Z.

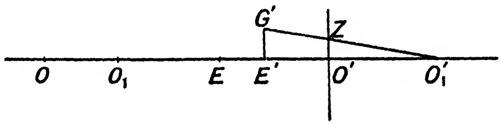

Fig. 11.2

Using the same notation as in the last example and writing $O'Z = z$, we have, from the similarity of the triangles $G'O_1'E'$ and $ZO_1'O'$,

$$\frac{a'}{p_1'} = \frac{z}{p_1' - p'},$$

or

$$z = a'\left(1 - \frac{p'}{p_1'}\right) = a' \cdot \frac{Q' - Q_1'}{Q'}.$$

If z be expressed in terms of quantities in the object space, this becomes

$$z = a' \frac{Q_1 - Q}{a'aD - Q}. \qquad . \quad . \quad . \quad 11.8$$

The eye in perceiving an image notices this blurring effect only if the radius of the circles is greater than a certain amount z_0. If the point O_1' lie to the left of the screen at O′ the value of z is of the opposite sign to that in (11.8). Thus if z lies between the values $\pm z_0$, the

eye will perceive a sharp image. This corresponds to a displacement of O_1' from a certain distance in front of the screen at O' to a certain distance behind it, and hence to a certain range of position for O_1. Thus to the eye there is a certain distance along the axis within which sharp images can be formed. We have what is known as depth of focus, and this corresponds to values of Q_1 lying between

$$Q + \frac{a'aD - Q}{a'} z_0 \text{ and } Q - \frac{a'aD - Q}{a'} z_0. \qquad 11.9$$

The Eye

The eye is a complicated refracting system consisting from before backward of the cornea, aqueous humour, crystalline lens and vitreous humour. It is provided with an adjustable aperture, the pupil, and with a screen, the retina. The pupil lies immediately in front of the lens, and the retina is on the wall of the posterior chamber which contains the vitreous humour.

The entrance pupil of the eye is the image of the pupil of the eye in the part in front of the lens, and the exit pupil is the image of the pupil in the lens and vitreous humour. The whole system is thus a complex optical system, and we shall use the notation adopted in the previous chapters to describe it. The surfaces of the lens can be made more or less curved by muscular action, with the result that its focal length may be varied, giving the eye the power of accommodation.

The point at which the axis of the eye cuts the retina is described as the retinal point. When the eye is at rest the point conjugate to the retinal point is called the far point. In normal eyes the far point is at infinity, and to describe this property the eye is said to be emmetropic. When the far point is not at infinity, the eye is ametropic, and we describe the eye as myopic if the far point is at a finite distance from the front of the eye, and as hypermetropic if behind it. Thus, in the myopic eye a diver-

gent pencil of rays is brought to a focus on the retina and in the hypermetropic eye a convergent pencil is focussed on the retina without accommodation.

When the eye has the greatest power, the lens being of the greatest convexity, the conjugate of the retinal point is called the near point. The distance of the near point increases with age, varying from about 7 cm. in childhood to 70 cm. in later life, but at middle age it is in the neighbourhood of 25 cm., and it is customary to regard this as an average value in many calculations on eye problems.

The object focal distance of the eye is about 15·5 mm. when the eye is focussed on the far point, and about 14 mm. when focussed on the near point.

The angle of vision is the angle subtended by an object at the eye, and in the case of near objects it becomes of importance at which point the vertex of the angle is made. We shall take either the object focal point or the first principal point. Writing θ_F and θ_H for these angles, we have

$$\theta_F = \frac{y}{x}, \quad \theta_H = \frac{y}{l}.$$

But

$$\frac{y'}{y} = -\frac{f}{x} \quad (6.21),$$

hence

$$y' = -f\theta_F. \quad \quad . \quad . \quad . \quad . \quad . \quad 11.10$$

If we wish to take the principal point as the apex of the angle, we must apply equations (6.19) and (6.23), in which we write $n = 1$, since the first medium is air and n' is the refractive index of the vitreous humour. We have

$$y' = -\frac{l'}{n'} \cdot \frac{y}{l} = -\frac{l'}{n'}\theta_H. \quad \quad . \quad . \quad . \quad 11.11$$

In this formula l' is the distance between the image principal point and the retina, and it is possible to make

the assumption that the position of this principal point varies very little during accommodation. Thus for every condition of accommodation the eye produces retinal images of the same size for all objects which subtend the same angle at the object principal point. We may similarly use the formula (11.5) if we wish to place the apex of the angle subtended by the object at the centre of the entrance pupil.

The pupil of the eye and consequently the entrance and exit pupils vary in size during accommodation, and they vary also to some extent in position.

When focussed on an infinitely distant point, the ratio of the radius a' of the exit pupil to a that of the entrance pupil is $\frac{a'}{a} = 0{\cdot}923$. When the eye is focussed on the near point, the ratio is 0·941. The average value of a may be taken to be 2 mm.

The smallest object that can be distinguished by the eye as an object possessing extent subtends an angle at the eye known as the limiting angle of resolution. If, in any case, the angle subtended be less than this, the effect on the retina is such that the object is described as a point. The value usually taken for this angle is one minute of arc or 0·000291 radians, although recent investigations suggest that the eye can resolve much smaller angles. We shall adopt this value in order to illustrate the method of determining the depth of focus of the eye.

From (11.10) it follows that the diameter of the illuminated disc in the retina from an object subtending this angle at the eye is of magnitude $15{\cdot}5 \times {\cdot}000291$ or 0·0045 mm., where we have taken the value 15·5 mm. for the focal length. Thus in the formula (11.9) the value of z_0 is one half of this, i.e. 0·00225 mm.

The value of $Q\left(=\frac{na^2}{p}\right)$ is small since we may regard the angle as subtended by a distant object, the eye being focussed on infinity.

Thus Q_1 lies between $\pm a\mathrm{D}z_0$, i.e. $\frac{na^2}{p_1}$ lies between these values, or p_1 lies between $\pm\frac{a}{\mathrm{D}z_0}$ since $n=1$.

On substituting $a=2$ mm., $f=15{\cdot}5$ mm., and remembering that $\mathrm{D}=\frac{n}{f}=\frac{1}{f}$, we obtain the result that p_1 must lie between the values ± 14 metres. Thus all objects at distances of more than 14 metres appear sharply focussed to a normal eye focussed on infinity. The meaning of the negative distance is that a pencil of rays converging to points at distances more than 14 metres behind the eye also appear as if sharply focussed on the retina.

The Combination of the Eye and a Thin Lens

This is an example of the combination of optical systems which we studied in Chapter VI. It is of great practical importance and we shall consider it in detail.

Let the lens have focal length f_0 and let the two focal lengths of the eye be f and f'.

Suppose that the lens is situated at distance d in front of the object principal point of the eye.

Then if F denote the focal length of the combined system, we have from (6.34)

$$\mathrm{F}=-\frac{f_0 f}{\Delta}$$

where

$$\Delta=d-f_0-f.$$

Writing

$$\mathrm{D}_c=\frac{1}{\mathrm{F}},\quad \mathrm{D}_0=\frac{1}{f_0},\quad \mathrm{D}=\frac{1}{f},$$

for the various systems, we have

$$\mathrm{D}_c=\frac{1}{f}+\frac{1}{f_0}-\frac{d}{f_0 f}=\mathrm{D}+\mathrm{D}_0-d\mathrm{D}\mathrm{D}_0. \qquad 11.12$$

Let h denote the distance of the object principal point of the combined system from the lens, then from (6.36),

$$h = \frac{f_0^2}{\Delta} + f_0 + \frac{f_0 f}{\Delta} = -\frac{d\text{D}}{\text{D}_c}, \quad . \quad 11.13$$

while, if h' denote the distance of the image principal point of the combined system from that of the eye, we have, by (6.37),

$$h' = \frac{ff'}{\Delta} + f' + \frac{f_0 f'}{\Delta} = \frac{df'}{\Delta} = \frac{n\,df}{\Delta},$$

since, if n is the refractive index of the vitreous humour, we have $\frac{1}{f} = \frac{n}{f'}$, and thus

$$h' = -\frac{n\,d\text{D}_0}{\text{D}_c}. \quad . \quad . \quad . \quad . \quad . \quad 11.14$$

We again draw attention to the sign convention in applying these formulæ (cf. p. 101).

If the lens is placed at the first focal point of the eye, $d = f$, and from (11.12) $\text{D}_c = \text{D}$, i.e. the power of the combination is equal to that of the unaided eye.

The values of h and h' become

$$h = -\frac{1}{\text{D}} = -f \quad . \quad . \quad . \quad 11.15$$

and

$$h' = -\frac{n\text{D}_0}{\text{D}^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad 11.16$$

The second focal length of the combination, say F′, is given by

$$\frac{1}{\text{F}} = \frac{n}{\text{F}'},$$

and since $\frac{1}{f} = \frac{n}{f'}$ and $\text{F} = f$, we have $\text{F}' = f'$.

The focal distances are unaltered by the addition of the thin lens in this case.

From (11.15) we can find the position of the object principal point of the combination, the equation showing that it coincides with that of the unaided eye. Thus if we take the value of f as 15 mm., the lens is placed 15 mm. in front of the principal point of the eye, say to the left of the eye, as in our representation in Chapter VI. Then $h = -15$ mm. means that the principal point lies 15 mm. to the right of the lens, i.e. at the principal point of the eye.

On the other hand, (11.16) shows a displacement of the image principal point to the left.

The first nodal point lies at a distance f' from the first focal point and f' remains unaltered in this case, so that the first nodal point as well as the object principal and focal points remain unaltered in position. The corresponding image points are, however, all displaced by an equal amount h'. We can express h' in terms of focal distances; since $\frac{n}{D^2} = ff'$ and $D_0 = \frac{1}{f_0}$, we obtain from (11.16),

$$h' = -\frac{ff'}{f_0}. \quad . \quad . \quad . \quad . \quad 11.17$$

Effect of a Lens on the Size of the Retinal Image

Let a distant object subtend an angle $\theta_H \left(= \frac{y}{l}\right)$ at the object principal point; the image is formed in the image focal plane ($l' = f'$), so that from (6.19) we have

$$y' = -f\theta_H, \quad . \quad . \quad . \quad . \quad . \quad 11.18$$

where f is the object focal length of the system.

In the case of the eye without a lens we have

$$y' = -\frac{\theta_H}{D}$$

and with the lens $y'' = -\dfrac{\theta_H}{D_c}$.

The ratio $\frac{y''}{y'}$ is the spectacle magnification, and it can be written

$$\frac{y''}{y'} = \frac{D}{D_c} = \frac{1}{1 - D_0(d - f)}.$$

$(d - f)$ measures the distance x of the lens from the first focal point of the eye, thus

$$\frac{y''}{y'} = \frac{1}{1 - xD_0}. \quad . \quad . \quad . \quad 11.19$$

The special case where the lens is at the object focal point corresponds to $x = 0$, and the spectacle magnification is unity.

Correction of Ametropia

The example of the combination of the lens and eye with the former at the eye's object focal point has an interesting application.

In fig. 11.3 let H_2' denote the image principal point of a short-sighted (myopic) eye. Let R denote the retinal

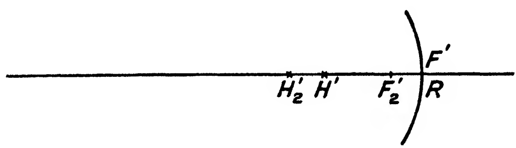

Fig. 11.3

point of the eye. In this case the image of an infinitely distant object falls in front of the retina, i.e. the image focal point is at a position such as F_2', where $H_2'F_2'$ is f' in the notation we have just used.

The eye we are considering has an eyeball which is too long by $F_2'R$ ($=\delta$, say). Now if we place a lens of focal length f_0 at the object principal point of the eye, the image focal length remains the same but the position of H_2' is displaced by an amount $-\frac{ff'}{f_0}$. If we can arrange

that this displaces H_2' to H', where $H_2'H' = F_2'R = \delta$, the image focal point of the combination will lie at R. In this case the image of a distant object will be thrown on the retina, and the combination is emmetropic. We have thus to choose f_0 so that

$$\delta = -\frac{ff'}{f_0}.$$

In the case we have considered, since f and f' are both positive, f_0 is negative, or we require a concave lens for the correction of the myopic eye.

The Eye in Combination with an Optical Instrument. Effect on Size of a Retinal Image

Let an optical instrument have its entrance and exit pupils situated at E and E′ and let F_1' be the image principal focus (fig. 11.4). Let the object principal focus of the eye be situated at F_2 at a distance e from the exit

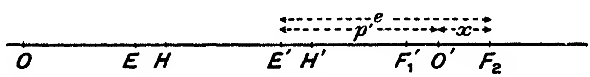

Fig. 11.4

pupil of the instrument, and let O′ be the position of the image formed by the instrument. Let an object of length y be seen by the eye from some conventional distance, and let the object in this case be at a distance d_0 from F_2. If, as before, we denote the object focal length of the eye at rest by f, we obtain for the length of the retinal image

$$y_0' = -\frac{fy}{d_0} \quad \text{(cf. 6.21)}. \quad . \quad . \quad 11.20$$

Let y' be the length of the image of the same object produced by the instrument at O′. By (11.5)

$$y' = -\frac{anp'}{a'n'p}y, \quad . \quad . \quad . \quad 11.21$$

where n and n' are the refractive indices of the object and image spaces respectively.

Suppose that the eye sees this image and that it undergoes accommodation, the focal length changing from f to f_a.

The length of the image now seen on the retina is

$$y'' = -\frac{f_a y'}{e - p'}. \qquad \ldots \quad 11.22$$

Let us examine the ratio $\frac{y''}{y_0'}$.

From the three equations just obtained it follows that

$$\frac{y''}{y_0'} = -\frac{n}{n'}\frac{a}{a'}\frac{p'}{p}\frac{f_a}{f}\frac{d_0}{e - p'}. \qquad \ldots \quad 11.23$$

If reference be made to the principal points of the instrument, e being the distance $H'F_2$ instead of $E'F_2$, we make use of (6.19) instead of (11.21), i.e. of

$$y' = -\frac{f_1}{f_1'}\frac{l'}{l}y = -\frac{nl'}{n'l}y, \qquad \ldots \quad 11.24$$

where f_1, f_1' refer to the instrument, and in (11.22) we write l' instead of p'. It follows that

$$\frac{y''}{y_0'} = -\frac{n}{n'}\frac{l'}{l}\frac{f_a}{f}\frac{d_0}{e - l'}. \qquad \ldots \quad 11.25$$

These formulæ do not hold for the telescope since the instrument formulæ differ from those used above, but they are true for other optical instruments.

Some optical instruments are so constructed that in the normal eye at rest sharp images are formed. This corresponds in the above formula to $p' = \infty$, since the images in the instruments appear very distant. The eye is usually situated in air so that $n' = 1$, and in this case there is no accommodation $f_a = f$.

Equation (11.23) becomes

$$\frac{y''}{y_0'} = n\frac{a}{a'}\frac{d_0}{p}. \qquad . \quad . \quad . \qquad 11.26$$

Moreover, from (11.2) we obtain in this case

$$\frac{af_1}{a'p} = 1 \quad \text{or} \quad \frac{a}{a'p} = \frac{D_1}{n}, \text{ where } D_1 = \frac{n}{f_1}.$$

Thus

$$\frac{y''}{y_0'} = d_0 D_1. \qquad . \quad . \quad . \quad . \qquad 11.27$$

Let the radius of the entrance pupil subtend an angle θ at the object point O, then for small angles $\theta = \dfrac{a}{p}$.

We write $\alpha = n\theta = \dfrac{na}{p}$ and call α the numerical aperture of the instrument. Since for a distant object, i.e. for a small value of θ, $\dfrac{a}{a'p} = \dfrac{D_1}{n}$, it follows that

$$\alpha = D_1 a'. \qquad . \quad . \quad . \quad . \qquad 11.28$$

The ratio $\dfrac{y''}{y_0'}$ is described as the magnification of the optical instrument, and we see that, in the simple case just considered, the magnification for the unaccommodated normal eye is equal to the product of the power of the system and the conventional distance d_0. It is not dependent on the position of the eye with regard to the exit pupil.

The Simple Magnifying Glass

The simple magnifying glass consists frequently of a chromatically corrected thin converging lens. There are other more complicated instruments, but we shall consider this simple type to illustrate the use of the magnifying glass.

This case is the same as the combination of the eye

and lens, but we shall consider it again particularly with regard to magnification, since this is the purpose in this case. The lens is placed in a mount, and its boundary is both entrance and exit pupil.

We have here a particular case of the application of formula (11.23), in which $a = a'$, $n = n' = 1$. Thus the retinal magnification is

$$m = -\frac{p'}{p}\frac{f_a}{f}\frac{d_0}{e - p'}, \quad . \quad . \quad . \quad 11.29$$

e is the distance between the lens and the object focal point of the eye, p is the distance of the object from the lens, and p' the distance of the image formed by the lens. If the lens is at the focal point of the eye, $e = 0$ and

$$m = \frac{d_0}{p}\frac{f_a}{f}.$$

If, however, the image is at the near point and if the standard position for comparison is taken to be at the near point, the eye undergoes no accommodation between the observation with the lens and that without it. Thus $f_a = f$ and $p' = -d_0$.

We have seen that a lens in the position of the object focal point of the eye causes no change in the size of the image on the retina (11.19), and here the magnification m is equal to that of the lens itself. Let the focal length of the lens be F, then

$$-\frac{1}{d_0} + \frac{1}{p} = \frac{1}{\mathrm{F}}$$

and

$$m = \left(1 + \frac{d_0}{\mathrm{F}}\right).$$

The average value of $d_0 = 25$ cm., so that

$$m = 1 + \frac{25}{\mathrm{F}}.$$

If we express the power in dioptres, i.e. $D = \frac{1}{F \text{(in metres)}}$, the unit of power being that of a convergent lens in air of focal length 1 metre,

$$m = 1 + \frac{D}{4}. \quad . \quad . \quad . \quad . \quad 11.30$$

The Compound Microscope

The equation (11.30) shows that, if we wish to obtain a great magnification with a single lens, we must choose one of great power, i.e. of short focal length. The curvatures of the surfaces of such a lens would be great and it would be relatively thick. Apart from the practical inconvenience of its use, it would have defects due to spherical aberration.

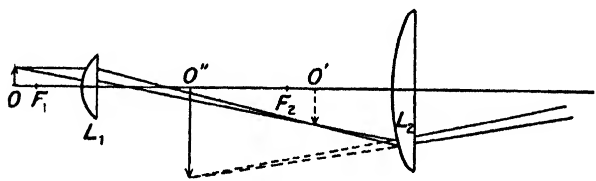

Fig. 11.5

By the use of two lenses we can obtain a greater magnification, and the deviation of the rays forming the image is shared over several surfaces.

The principle of the instrument is illustrated by fig. 11.5. The lens L_1 is the objective, and it forms an image O′ of an object O situated at a point a little farther away from the lens than the object focal point. The image is then magnified by the eye lens L_2, producing a final image at O″. The first image at O′ must lie nearer to L_2 than its object focal point.

The microscope is required to produce a large bright image of a very small object. If we refer to equations (9.3) and (9.15), we see that this aim necessitates a large

numerical aperture, for this ensures a high resolving power and a high value for the brightness of the image.

Thus the objective must have a large numerical aperture. It has a small field of view, but the rays entering it make large inclinations with the axis.

On the other hand, the eye-piece has a large field of view, but the rays are limited by the pupil of the eye; the pencils taking part in the formation of the final image are of small divergence. If we examine the figure (11.5), we see that with a single lens L_2 the pencils in the outer part of the field would not enter the pupil of an eye near the lens, and the field would thus be limited. We require, however, to see a large field, and this leads to an elaboration of the eye-piece.

The objective and the eye-lens have thus each their own special difficulties to overcome, and we will consider them separately.

Taking the objective first, we see that with the large inclinations of the rays there will be difficulties of spherical aberration. These must be eliminated, and to do this we satisfy the sine law, i.e. aplanatic condition (7.71).

In the consideration of spherical aberration we neglected the cubes of the inclinations of the rays to the axis. But in microscope objectives the angles amount to 60° or more, so that we have to go beyond the limits of the theory we have considered in attempting to eliminate this defect. This elimination is attained by making use of aplanatic foci and thus satisfying the sine condition. It is possible for spherical aberration to be over-corrected, and this over-correction may then be neutralized by the use of an under-corrected system. In this way spherical aberration may be almost completely eliminated.

We must also correct for chromatic effects. The aplanatic condition ought to be satisfied for at least two colours, as we have seen at the end of Chapter X. It is sometimes sufficient to obtain partial achromatism by the objective and make further correction by the eye-

piece. The system is made to produce the coloured images from an object near the principal focus in the same position. They are not, however, of the same size, and this has to be corrected by the eye-piece.

This is the principle employed in objectives with wide apertures. A single nearly hemispherical lens is used combined with strongly over-corrected systems of lenses.

We have seen how systems may be corrected to some extent for rays of two or three colours. But there is always a residual effect—the uncorrected colours producing a so-called secondary spectrum. In addition to this, we have to consider the chromatic effect of spherical aberration (end of Chapter X). Abbe has referred to this as the chromatic difference of spherical aberrations, and has shown that if a system is constructed to eliminate the spherical aberration for a mean refractive index, there is still a residual effect for other indices. He has shown that the defect may be overcome by placing lenses at a suitable distance apart.

In immersion systems the space between the front lens of the objective and the object is filled with an oil. The effect is to increase the numerical aperture of the instrument.

If the oil has the same refractive index and dispersion as the front lens, we have the so-called homogeneous immersion. In this case the principle considered on p. (135) can be used to obtain aplanatic images. We saw that when a small object is placed at a distance r/n from the centre of a sphere of refractive index n, all the rays emerging from the surface of the sphere appear to come from an object at distance nr from the centre.

In fig. 11.6 let L_1 denote the first lens with C as the centre of curvature of the posterior surface. With the homogeneous immersion we have a homogeneous medium to the left of this surface, and if the object lie at P_1, where $CP_1 = r/n$, the rays emerging appear to come from an object at P_2, where $CP_2 = nr$.

Let a second lens be placed so that its first surface has

its centre of curvature at P_2. Let its back surface have radius r_2, and suppose that P_2 is an aplanatic point for this surface. The image now will appear to lie at a point P_3, and the process may be continued. In this way the rays are gradually converged and the image is without aberration.

We must not forget that in this way large chromatic errors are introduced, and in practice only the first two lenses of the objective are constructed according to this principle. Otherwise it is impossible to correct the resulting colour effects.

As a result of the work of Abbe the microscope objective has been brought to a high state of perfection. He has designed the apochromat (see end of Chapter X), which satisfies the requirements outlined. It consists of a combination of ten lenses, and it is used with a homogeneous immersion. It is achromatic for three colours, and the aplanatic conditions are satisfied for two. The focal length is 2 mm., and the numerical aperture 1·40 (fig. 11.7).

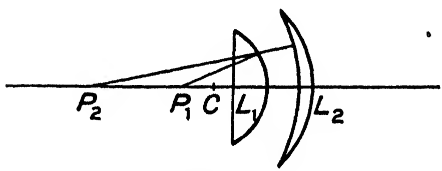

Fig. 11.6

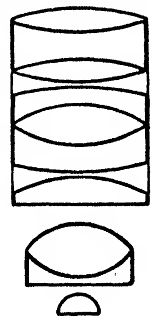
Fig. 11.7

The eye-piece has to eliminate astigmatism from the pencils falling obliquely upon it, to eliminate colour defects and to form orthoscopic images (p. 169).

We have already mentioned one difficulty in connection with the use of a single lens as eye-piece.

In practice two lenses are used, the one nearer the objective being called the field lens because it increases the field of view of the instrument, and the one nearer the eye, the eye lens.

In the case of the apochromat objective we saw that the sizes of the differently coloured images were not the same. This remaining defect of achromatism can be corrected by using an eye-piece of the opposite achromatism. In the apochromat the blue rays form the larger image, and the eye-piece combined with it produces a red image larger than the blue. If, however, the achromatism of the object glass is not troublesome, the eye-piece is made as closely as possible to have the same focal length for different colours. When it is not possible to adjust two lenses to satisfy the condition (10.14) exactly, an approximation is made. Suppose that the eye-piece is achromatic with regard to the focal length, and let a ray of white light fall upon it. This ray may be regarded as a bundle of parallel coloured rays. By (6.29) they will all emerge at the same inclination θ to the axis, since f is the same for them all. Thus if the emergent rays enter the eye focussed for infinity, they are reunited and the image seen is colourless. Even when the eye is focussed for the nearest distance of distinct vision, an eye-piece so corrected introduces little colour defect.

The use of four surfaces instead of two, by replacing the single lens by an eye-piece of this kind, divides the labour of deviation and improves the system with regard to spherical aberration and also with regard to astigmatism.

In addition to these systems of lenses, an arrangement of lenses somewhat similar to that of the objective, but in the reverse order, is placed below the stage on which the object is carried. This is the condenser, and its function is to illuminate the object, providing rays of wide divergence in order that the full numerical aperture may be used.

The Magnification of a Microscope

In fig. 11.8 let H_1 and H_1' denote the principal points of the objective, and H_2 and H_2' those of the eye-piece.

An object of length y gives rise to an image y' formed by the objective which lies close to the field lens. The final image is y'' at a distance δ from the eye.

Let f_1 and f_1' denote the focal lengths of the objective. These are not equal in immersion systems. Let f_2 denote the focal lengths of the eye-piece since in this case both are equal.

The distance from field lens to objective is almost equal to the length L of the microscope, and we shall also take $F_1'O' = L$. We shall also regard $F_2'O''$ as equal

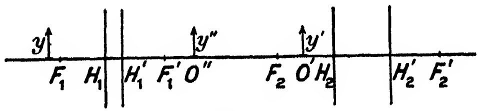

Fig. 11.8

to δ. Using the formulæ with respect to the focal points (6.21),

$$\frac{y'}{y} = -\frac{L}{f_1'}, \quad \frac{y''}{y'} = -\frac{\delta}{f_2}.$$

Thus

$$\frac{y''}{y} = \frac{L\delta}{f_1'f_2}. \qquad \ldots\ldots \quad 11.31$$

By (6.33),

$$f' = -\frac{f_1'f_2}{\Delta},$$

and Δ may be put equal to L in this case.

Thus

$$\frac{y''}{y} = -\frac{\delta}{f'}, \qquad \ldots\ldots \quad 11.32$$

where f' is the focal length of the microscope considered as a whole.

Examples of Eye-pieces

We shall describe two eye-pieces in common use; the first is Huyghens' and the second Ramsden's.

Huyghens' eye-piece consists of two separated simple plano-convex lenses with the curved surfaces turned towards the incident light.

We have seen that such a combination has the same focal length for all colours when the lenses are made of the same material and when the separation is equal to half the sum of the focal lengths.

This condition is satisfied in this eye-piece, so that with the notation of (10.14) $a = \frac{1}{2}(f_1 + f_2)$.

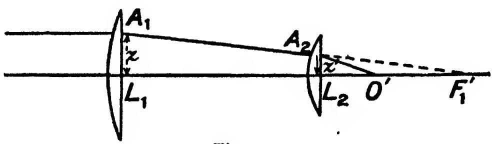

Fig. 11.9

Errors due to spherical aberration are reduced by allowing each lens to share equally in the deviation. The condition imposed to bring about this result is to place the lenses at a distance $(f_1 - f_2)$ apart. In order to see that this is the case, in fig. 11.9 let L_1 and L_2 denote the two lenses.

Let a ray originally parallel to the axis strike the field lens at A_1 and, after refraction, let it strike the eye lens at A_2.

From the figure it is seen that the deviation by L_1 is $\angle A_1F_1'L_1$. The deviation by L_2 is $\angle O'A_2F_1'$, and if the refraction is equally shared, this must be equal to $\angle A_1F_1'L_1$ or $A_2O' = O'F_1'$. For small angles $O'A_2 = O'L_2$ or O' is midway between L_2 and F_1'.

But O' and F_1' are conjugate points for L_2, and thus

$$\frac{1}{L_2O'} - \frac{1}{L_2F_1'} = \frac{1}{f_2},$$

and since $L_2O' = \frac{1}{2}L_2F_1'$, it follows that $L_2F_1' = f_2$. Thus

$$L_1L_2 = a = f_1 - f_2. \quad . \quad . \quad . \quad 11.33$$

This condition combined with that for achromatism requires

$$f_1 = 3f_2, \quad a = 2f_2. \quad . \quad . \quad . \quad 11.34$$

This condition is independent of the form of the lenses, although in practice plano-convex lenses are used. The aberrations depend on the forms, and the different defects sometimes require different forms for their correction. In practice a compromise is arrived at.

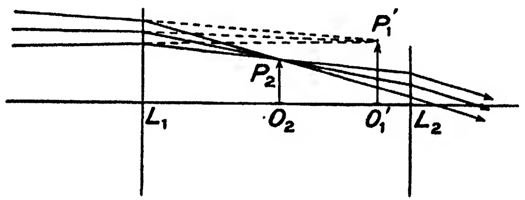

Fig. 11.10

Let L_1, L_2 in fig. 11.10 denote the positions of the lenses, and suppose the rays from the objective to be directed towards an image $O_1'P_1'$. They will be deviated by the field lens to form an image at O_2P_2, and we shall suppose that the rays emerge parallel from the eye lens. Thus $O_2L_2 = f_2$, and it follows that $L_1O_2 = f_2$ and $L_1O_1' = \frac{3}{2}f_2 = \frac{3}{4}L_1L_2$. Suppose that it is required to make measurements upon the image in this instrument. The natural procedure would be to place a scale or cross at the position of the image formed by the objective, and view both together by the eye-piece. This is not possible with Huyghens' eye-piece, since the image formed by the action of the field lens falls behind this lens. Thus, if a cross wire is placed in the position of the image, while the latter undergoes refraction in both lenses of the eye-piece, the former is viewed by the eye

lens alone. This gives rise to unequal distortions in the image and cross wire, and measurements are rendered untrustworthy.

In Ramsden's eye-piece the lenses are of equal focal length, so that the achromatic condition requires a separation equal to the focal length. This has the disadvantage that small objects like dust particles or scratches on the surface of the field lens lie almost at the principal focus of the eye lens, and are thus much magnified and disturb the field of view. This difficulty is overcome by placing the lenses somewhat closer together, and in practice they are at a distance apart equal to two-thirds

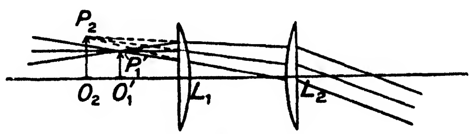

Fig. 11.11

of the focal length. The field lens presents its plane surface to the incident rays, and the eye lens its curved surface.

Although the achromatic condition is not satisfied, the resulting defect is not unduly disturbing, and it can be eliminated by making both lenses an achromatic pair in contact. Again, the spherical aberration is diminished by the principle of division of refraction between several surfaces. Moreover, the image is fairly free from distortion.

The path of the rays is illustrated in fig. 11.11. In this case a real image is formed by the objective at $O_1'P_1'$, and the field lens forms a virtual image of it at O_2P_2. If the rays emerging from the eye lens are parallel, it may be shown that the distance L_1O_1' is $\frac{1}{4}f$, where f denotes the focal length of either lens. Thus, if a cross wire or scale be placed at $O_1'P_1'$, both it and the image are able to be examined by the action of the whole eye-

piece, and satisfactory measurements may be made by this means.

The Resolving Power of the Microscope

We saw in the discussion on the eye that two rays entering it could be recognized as separate rays if they were inclined at not less than one minute of arc.

In the microscope the objective does not form an absolutely flat image, but the image is made up of small discs corresponding to the points of the object (p. 167). If two points of the object are to be seen as distinct, it is necessary that the corresponding discs should not overlap. If they do, the eye-piece will not separate them. Provided that there is no overlapping, the eye will observe the discs as separate, if the rays from them enter the eye at an inclination not less than one minute of arc.

Let us suppose that the objective is perfect, i.e. two points O_1 and O_2 of the object give rise to two points O_1' and O_2' in the image formed by it. Let us suppose that the object is illuminated by light of wave-length 6000 Å. From (9.3) it follows that the smallest separation of O_1O_2 compatible with resolution is

$$O_1O_2 = \frac{\lambda}{2a},$$

where a is the numerical aperture.

Suppose that a has the value 1·5. (The limit reached for numerical apertures is somewhat higher than this.) Thus

$$O_1O_2 = \cdot 0002 \text{ mm.}$$

If the distance of the final image from the eye be 25 cm., the points corresponding to O_1O_2 must subtend $1'$, i.e. they must be separated by a distance of ·071 cm. Thus the magnification necessary to obtain the limit of resolution is 355. This takes no account of the imperfections of the objective, which would require a greater

separation O_1O_2 than the limit, according to (9.3). Thus the magnification obtained when the image points of O_1 and O_2 subtend 1′ at the eye is less than that calculated.

From (9.15) the ratio of the brightness of the image to the natural brightness can be calculated. If we suppose that the exit pupil of the instrument is less than the pupil of the eye, and that the latter has a diameter of 2 mm., the value is 0·35.

The Measure of the Aperture of the Microscope

In the cases where the exit pupil is not greater than the pupil of the eye, we have, from (9.14) and (9.15),

$$\frac{E^2}{p^2} = \frac{a^2}{m^2}\frac{d^2}{p^2}, \qquad 11.35$$

where E is the radius of the exit pupil. Thus

$$a = \frac{mE}{d}. \qquad 11.36$$

This gives a numerically, the numerical value of m being substituted in the formula.

Thus from (11.32) the value of a is given by

$$a = \frac{E}{f'} \qquad 11.37$$

or the numerical aperture is equal to the radius of the exit pupil divided by the image focal length of the whole microscope. This formula indicates how the value of a may be determined experimentally.

Finally, we come to the placing of stops in the microscope. Frequently the rim of the hemispherical lens of the objective is the entrance pupil or aperture stop. If the exit pupil is less than the pupil of the eye, this rim is actually the entrance pupil, but if the exit pupil is greater, the image of the pupil of the eye by the whole instrument acts as entrance pupil.

The other important stop is the field of view stop, which is placed, in the case of Huyghens' eye-piece, between the field and eye lenses of the eye-piece in the plane of the image formed by the objective and field lens. In this way the final image is made to be of uniform brightness up to its outer borders (p. 198), and the image formed by the field lens is arranged to fit the aperture of the field of view stop. When Ramsden's eye-piece is used, the field of view stop is placed at the position of the image formed by the objective.

The final image is either at infinity when the eye is not accommodated, or at the so-called distance of distinct vision when the eye is focussed for this distance.

The Telescope

The essential parts of the telescope are, as in the microscope, the objective and the eye-piece. The former is a convergent system producing an image of a distant

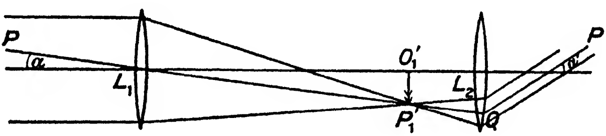

Fig. 11.12

object at its principal focus. The eye-piece enlarges the image, and when focussed for parallel rays the image formed by the objective lies in its object focal plane.

We have in this case an example of a telescopic system which we discussed at the end of Chapter VI.

Referring to equations (6.51) and (6.52), we see that the linear and angular magnifications are both constant. Equation (6.55) shows that these magnifications are reciprocals of one another.

In the astronomical telescope both objective and eye-piece are convergent systems, and $O_1'P_1'$ (fig. 11.12) is the image formed by the objective, the final image being

seen by means of parallel rays emerging from the eye-piece when the eye is at rest.

If f_1 and f_2 denote the focal lengths of the lenses,

$$L_1O_1' = f_1, \quad L_2O_1' = f_2.$$

Let two conjugate rays PL_1 and QP' cut the axis in α and α', then the angular magnification $m = \dfrac{\tan \alpha'}{\tan \alpha}$.

If we think of a ray drawn from P_1' to L_2, it would emerge from the lens parallel to QP', and thus

$$\tan \alpha' = \frac{P_1'O_1'}{O_1'L_2}.$$

Thus

$$m = \frac{L_1O_1'}{L_2O_1'} = \frac{f_1}{f_2}. \quad . \quad . \quad . \quad 11.38$$

The rim of the objective is generally the entrance pupil and the exit pupil is the image of this ring in the instrument. This exit pupil can be observed as a bright patch and its diameter measured. We have thus an image and object, the exit pupil or eye ring, as it is called in this case, and the rim of the objective. The linear dimensions of these are in the ratio of the linear magnification of the instrument. This quantity is a constant for all conjugate pairs, and if the eye ring have diameter E and the object glass rim diameter R, then by (6.55)

$$\frac{E}{R} = \frac{\tan \alpha}{\tan \alpha'} = \frac{f_2}{f_1} \text{ (numerically)}. \quad . \quad 11.39$$

The object glass is usually made of an achromatic pair of lenses placed close together.

The combination must have a given focal length F; thus

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2};$$

since it is achromatic we have the further condition

$$\frac{\omega_1}{f_1} + \frac{\omega_2}{f_2} = 0.$$

These determine the focal lengths, and we cannot satisfy any further condition between them.

Possible defects of the system are those of distortion, of astigmatism and of spherical aberration.

Distortion is absent since the rays are limited to a fairly small central aperture.

In order to eliminate the effect of astigmatism we have to arrange that the astigmatic difference of the two lenses is zero. This requires that the right-hand side of an equation like (7.59) must vanish. In the form given the focal length is not contained explicitly, but an examination of the equation shows that it involves a relation between the focal lengths.

We cannot, with the system of lenses used, impose this further condition, and thus some astigmatism remains.

Equation (7.64) gives the condition for absence of curvature. Applied to this case, we have to satisfy

$$\frac{1}{n_1 f_1} + \frac{1}{n_2 f_2} = 0.$$

The diminution of spherical aberration can be brought about by adjusting the curvatures of the surfaces. It is usual to cement the adjacent surfaces together, one being concave and the other convex of equal curvature. Equation (7.41) gives the spherical aberration for one lens, and by the combination it is found possible to reduce the total aberration of the two lenses to zero. This imposes a condition on the curvatures of the lens surfaces, but the condition can be satisfied in addition to the two conditions relating to the focal length. In fact, all the conditions taken together serve to determine the curvatures of the lens surfaces.

Brightness of the Image

From (9.13) we see that if E denote the radius of the eye ring and p that of the pupil of the eye, providing that $E < p$,

$$\frac{H}{H_0} = \frac{E^2}{p^2},$$

and if R is the radius of the object glass, since $E = \frac{R}{m}$, by (11.38) and (11.39),

$$\frac{H}{H_0} = \frac{R^2}{m^2p^2}. \quad . \quad . \quad . \quad . \quad 11.40$$

This shows how the brightness depends upon the aperture of the object glass and the magnification.

Resolving Power

Similar remarks apply to the resolution of two points in this case, as in the case of the microscope.

The formula (9.2) gives the value of the angle subtended at the object glass which can just be resolved. Suppose that in going through the telescope a ray inclined to the axis at this limiting angle emerges at an inclination θ'. Then

$$\theta' = 1{\cdot}22\,\frac{\lambda}{R}m.$$

If $\theta' > 1$ min. of arc, i.e. $> {\cdot}000283$ radians, the eye will be able to appreciate that the ray is inclined to the axis. This is the case if $m > \frac{R}{1{\cdot}22\lambda} \times {\cdot}000283$.

When m has reached the limit $\frac{R}{1{\cdot}22\lambda} \times {\cdot}000283$, no advantage results in relation to resolving power, although it may be an advantage in the examination of an object.

Eye-pieces

Both Huyghens' and Ramsden's eye-pieces are used in telescopes as well as in microscopes, and in both cases the principal rays falling on the eye-piece are almost parallel to the axis. Similar remarks apply to the eye-pieces here as in the microscope. Ramsden's eye-piece finds more frequent application, since it is suitable for measurements as in spectrometers.

Galileo's Telescope

By replacing the convergent eye-piece of the astronomical telescope by a divergent one, we obtain the instrument known as Galileo's telescope, which is the form of telescope used in the opera glass. The most

Fig. 11.13

obvious difference between the two forms of telescope is in the length, for whereas the length in the astronomical telescope is equal to the sum of the focal lengths of the object glass and eye-piece, in the present case it is equal to the difference. For in order that the system shall be telescopic, the image focal point of the objective must coincide with the object focal point of the eye-piece, or the arrangement of fig. 11.13 is necessary.

Thus OE $= (f_1 - f_2)$, where f_1 and f_2 denote the focal distances of the objective and eye-piece respectively.

The formula (11.39) holds in this case, and here also the rim of the objective acts as the entrance pupil. The eye-piece forms a virtual diminished image of this rim in front of itself, i.e. the exit pupil lies in front of the eye-piece, and the pupil of the observer cannot be made to coincide with it.

The final image formed by the instrument is erect,

since the inverted image formed by the object glass is again inverted by the divergent eye-piece.

The Camera Lens

Photographic objectives are required to produce real images on a sensitized plate. The field of vision must be large, and since it is necessary that a bright image should be formed, the pencils of rays are wide-angled. The design of the objective depends upon the purpose to which it is to be put, whether for landscape, portrait or other work, and the degree of approach to optical perfection which is necessary depends upon the purpose.

It has been found possible to correct photographic systems for spherical and chromatic aberrations and for astigmatism. They can be designed also to fulfil the sine condition and to produce a flat image.

There are two periods in the construction of lenses, the earlier associated with the name of Fraunhofer and the later with the names of Abbe and Schott. In the earlier period the only kinds of optical glass known were those for which the refractive index and the dispersive power increased together. No two glasses were known for which the refractive index of one was greater than that of the other, and at the same time the dispersive power of the one was less than that of the other. This period ended with the production of glasses for which this relation is reversed.

The importance of this can be appreciated by reference to the condition for achromatism (10.11), i.e.

$$\frac{\omega_1}{f_1} + \frac{\omega_2}{f_2} = 0,$$

which must be satisfied by two lenses in contact, and to the condition (7.65)

$$n_1 f_1 + n_2 f_2 = 0,$$

which must be satisfied by two lenses when a flat image is required.

In the earlier period it was impossible to satisfy both. To this period certain cemented lenses belong which were corrected for spherical and chromatic aberrations. These are described as old achromats to distinguish them from the later types made of the new kind of glass which are called new achromats. A combination of the two types is called an anastigmatic aplanat.

The Petzval Portrait Objective

In the earliest days of photography it was necessary to make long exposures of from two to twenty minutes on account of the slowness of the plates. Petzval studied the design of a lens with a view to shortening the exposure time, and in accordance with his calculations a system was produced with great power of transmission of light and especially suitable for enlarging purposes.

It consists of two separated parts each of which is made achromatic, and it is free from spherical aberration. This is removed to a higher degree that we have studied in Chapter VII, and this requires more radii of the lens surfaces to be at the designer's disposal. In order to remove some other faults which would appear if the lenses were all together, the separation into two parts is necessary.

The system is not astigmatic, but some degree of this defect is not seriously detrimental. The design is devoted to the correction of the centre of the field, but the separation of the parts means a diminution of the field of view, and the many surfaces each producing some reflection means a loss of brightness. Nevertheless its defects are not serious enough to destroy its advantages, and it is a good example of a portrait objective.

In connection with photographic objectives, a quantity known as the aperture ratio or relative aperture is of importance, since it determines the illumination of the image and consequently the " rapidity " of the lens. It

is measured by the ratio of the diameter of the entrance pupil to the focal length, and the illumination can be shown to be proportional to the square of this ratio.

Steinheil's Antiplanat

The defect of the type of objective just described is that the system is not astigmatic, and definition in the image diminishes as we leave the centre of the field. In order to overcome this defect Steinheil designed the system known as the antiplanat. He obtained a reduction of the defects due to the obliquity of the pencils by making the objective of two members, each with high but opposed aberrations. In this way he was able to diminish both astigmatism and curvature over a certain region, but outside that region the defects persist.

Symmetrical Double Objective

This is described in order to illustrate the use of a stop in bringing about the formation of a correct image (fig. 11.14). It is suitable for the photography of buildings where an orthoscopic system is necessary.

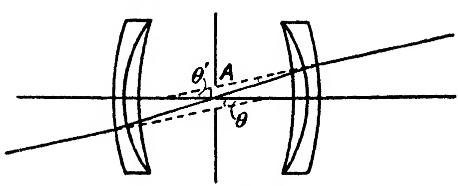

Fig. 11.14

Such a system can be satisfactorily corrected for colour effect as for spherical aberration. Astigmatism depends on the separation of the lenses and the image is not flat, but becomes flatter as the separation increases.

All principal rays pass through the centre of the stop A, and it is clear from the symmetry of the arrangement that an incident ray and the corresponding conjugate ray

are equally inclined to the axis. Thus the orthoscopic condition (7.67) is satisfied.

The Zeiss Anastigmat

It will be clear from the remarks at the beginning of this discussion on camera lenses that anastigmatic flattening was impossible in the earlier period. This condition changed in 1889 when Rudolph introduced an objective capable of giving great illumination and of producing a flat field.

The method adopted by Rudolph was to construct a system of the ordinary types of glass which was corrected for spherical and chromatic aberration, i.e. he made use of an old achromat. He constructed also a system of the new type of glass which was approximately corrected for these two aberrations, i.e. he made a new achromat. But both of these possessed the defect of astigmatism. This can only be eliminated by combining two systems which correct one another, and consequently their individual astigmatic errors are of opposite sign. At the same time the special property of the glass of the new achromat makes it possible to remove the curvature of the image, and by a suitable construction this flattening and the removal of astigmatism do not come into opposition with each other.

In this way an anastigmat was produced with a large aperture ratio and with uniform definition over a large area of the field.

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